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Separability and Non-Determinizability of WSTS

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Well-structured transition systems (WSTS) are among the most liberal transition systems that still admit decidability results. These are (typically infinite state) labeled transition systems (LTS), whose states are endowed with a well quasi order (WQO). The transitions of the WSTS must be compatible with this order, and if a state is final, then so must all the states that dominate it. Many popular models of computation, vector addition systems, lossy channel systems, and concurrent programs operating under weak memory models fall under the WSTS umbrella [4].

A recent separability result [3] for the languages of WSTS makes a surprisingly general statement: For two disjoint languages, respectively accepted by a deterministic and an unrestricted WSTS, there is a regular language that includes one language and completely excludes the other. The principal proof technique developed in [3] works for any order. However, the argument for ensuring a separator with finitely many states uses ideal decompositions of WQO's. Because the WSTS property is lost upon naive determinization, the determinicity assumption is hard to decouple from the argument. In this light, the separability result can be generalized to languages of all WSTS in one of two ways, none of which has lead to conclusions so far: (i) show that all WSTS languages can be accepted by deterministic WSTS, (ii) develop a new technique that is not based on ideal decompositions. Our first contribution is to develop a technique in line with (ii). Here, we employ a more subtle concept of limits, instead of ideal decompositions. Our second contribution is to show that (i) is not possible by giving a witness WSTS language that cannot be accepted by a deterministic WSTS.

1. Regular Separability

To show regular separability for all WSTS, we employ the proof principle developed in [3], which was also used to show the main result in [3]. Note that the proof principle refers to ULTS instead of WSTS. ULTS are LTS endowed with any (not necessarily WQO) order with which they are compatible. They form a superset of WSTS, because WSTS also require the endowed order to be a WQO.

Theorem 1.1 (Proof Principle for Regular Separability, [3]) *Given any two ULTS U and V, one deterministic with $L(U) \cap L(V) = \emptyset$, if there is a finitely represented inductive invariant S in $U \times V$, then the languages $L(U)$ and $L(V)$ are regularly separable.*

Inductive invariants are key to Theorem 1.1. An inductive invariant is a set of states that is (i) disjoint from the final states, (ii) contains the initial states, (iii) cannot be escaped by taking transitions. It is guaranteed to exist as soon as the language of the LTS is empty, which is the case for $L(U \times V) = L(U) \cap L(V)$. The challenging step in applying Theorem 1.1, is finding a *finitely represented* inductive invariant. Finite representation of S refers to the existence of a finite set $X \subseteq Q_\times$ with $S = \downarrow X := \{q \in Q_\times \mid q \leq p \in X\}$, where $(Q_\times, \leq)$ refers to the (ordered) states of the product ULTS. For Theorem 1.1 to apply, this challenge must be overcome in the setting of deterministic systems. Any ULTS can be determinized by moving on to the downward closed subsets of the original state space, ordered by inclusion. However, this determinization is not guaranteed to preserve the WQO property. Even though this is the case, a seldom used, weaker property must still hold. First observed by Rado [5], this property states that for any sequence of downward closed sets of states $[X_i]_{i\in\mathbb{N}}$, there is a convergent subsequence $[X_{\varphi(i)}]_{i\in\mathbb{N}}$ in the following sense. Any element p that appears in any set $X_{\varphi(i)}$, also appears in all but finitely many of the other sets $X_{\varphi(j)}$. A lattice-theoretic description of this property allows us to abstract away from the membership relation.

Definition 1.2 *A* converging lattice $(Q, \leq)$ *is a completely distributive lattice, where every sequence* $[p_i]_{i\in\mathbb{N}}$ *has a converging subsequence* $[p_{\varphi(i)}]_{i\in\mathbb{N}}$*. A* converging sequence $[q_i]_{i\in\mathbb{N}}$ *is an infinite sequence with*

$$\bigsqcup_{i\in\mathbb{N}} \bigsqcap_{j\geq i} q_j = \bigsqcup_{i\in\mathbb{N}} q_i \ .$$

Our approach is to initially determinize both WSTS and to find a finitely represented inductive invariant in the product by relying on convergence. We show that converging sequences $[q_i]_{i\in\mathbb{N}}$ and their limits are stable under transitions. Furthermore, we argue that if the limit is in the final states, then so must be an element from the sequence. Then, including the limits of all the converging sequences in a given inductive invariant S also results in an inductive invariant, $cl(S)$. This is the precise process that gives us the finite representation. We show that $cl(S)$ is chain complete (under the assumption of a countable state space), because all increasing sequences of subsets are convergent wrt. Definition 1.2. In this case, we can apply Zorn's Lemma to get maximal elements which represent the inductive invariant. Finally, we observe that there can only be finitely many maximal elements. Supposing there were infinitely many maximal elements, we see that a converging sequence could be extracted from these elements. This leads to comparability among maximal elements, which is a contradiction.

2. Non-Determinizability of WSTS

One way of getting rid of the determinicity assumption in [3] would be to show that all WSTS can be determinized. We show that this is not possible by constructing a WSTS language T that no deterministic WSTS accepts. To prove this, we employ a novel characterization of deterministic WSTS languages. This relies on a classical concept in formal languages, the Nerode quasi order. For a language $L \subseteq \Sigma^*$, the Nerode quasi order $w \leq_L v$ holds for $w, v \in \Sigma^*$, if $w.u \in L$ implies $v.u \in L$ for all $u \in \Sigma^*$. By a similar approach to the Myhill-Nerode Theorem, the characterization says that deterministic WSTS languages are precisely the languages whose

Nerode quasi order is a WQO. This is in contrast to the folklore result [1, Proposition 5.1] that says a language is regular if and only if the syntactic quasi order is a WQO.

Lemma 2.1 (Characterization of L(detWSTS)) *$L \in L(detWSTS)$ iff $\leq_L$ is a WQO.*

The state space of the WSTS that accepts our witness language T is the Rado structure $(R, \leq_R)$. This is a structure that is particularly suited for this task. Any WQO that loses the WQO property upon powerset construction embeds this WQO [2]. Using the Rado structure as our state space, we construct a (non-deterministic) WSTS that accepts the language $T \subseteq \{a, \bar{a}, zero\}^*$ with the property

$$T \cap a^*.\bar{a}^*.zero^* = \{a^n.\bar{a}^n.zero^i \mid i \in \mathbb{N}\} \cup \{a^n.\bar{a}^k.zero^i \mid i \in \mathbb{N}, n-k > i\}.$$

We can already deduce that $\leq_T$ is not a WQO from this description. The order contains the infinite antichain $[a^i]_{i\in\mathbb{N}}$. Assume $n > k$. For $a^n \not\leq_T a^k$, we have $a^n.\bar{a}^n \in T$ while $a^k.\bar{a}^n \notin T$. Conversely, for $a^k \not\leq_T a^n$, we have $a^n.\bar{a}^k.zero^{n-k} \notin T$ while $a^k.\bar{a}^k.zero^{n-k} \in T$.

3. Further results

If the states of an ULTS are ordered by a reversed WQO, the ULTS is called a downward-WSTS (DWSTS). By slightly modifying our proofs from the previous sections, we also deduce results for the class of languages accepted by DWSTS. First we note that the languages of DWSTS are those of WSTS with the words reversed. Combining this with the closure of regular languages under reversal yields the regular separability of disjoint DWSTS languages. We also observe that reversing the transitions in the WSTS that accepts T results in a deterministic DWSTS. This insight gives us the remaining relations between the language classes, summarized in Figure 1.

We shortly clarify the relations depicted in Figure 1. Reversing the languages of deterministic WSTS might not result in deterministic DWSTS languages, and vice-versa. For both classes WSTS and DWSTS, non-determinism results in a strictly more expressive class of langauges. Finally, the languages of deterministic WSTS are exactly the complements of the deterministic DWSTS languages, and the languages of WSTS are exactly the revesals of the languages of DWSTS.

$$\begin{array}{ccc} L(\text{detWSTS}) & \xrightarrow{\subsetneq} & L(\text{WSTS}) \\ {\scriptstyle \not\subseteq_{rev},\, \not\supseteq_{rev},\, =_{cmp}} \updownarrow & & \updownarrow {\scriptstyle =_{rev}} \\ L(\text{detDWSTS}) & \xrightarrow{\subsetneq} & L(\text{DWSTS}) \end{array}$$

Figure 1: Relations between language classes.

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