Infinite Nyldon Words

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Factorising the free Monoid

Theorem (Chen, Fox, Lyndon (1954))

$\forall w \in \Sigma^* \quad \exists k \in \mathbb{N} \quad \exists \ell_1, \ldots, \ell_k \in \mathcal{L}$ *unique*

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Corollary (Charlier et al. (2019))

$w \in \mathcal{L} \iff w \in \Sigma$

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Corollary (Charlier et al. (2019))

$w \in \mathcal{L} \iff w \in \Sigma \quad \vee$ *no factorisation with at least two factors*

Adjusting the Definition

Definition (Grinberg (2014))

1) $w \in \Sigma$, or

w is *Nyldon* : $\Longleftrightarrow$

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Example

0	✔		
1	✔		

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Example

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1	✔		
10			

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Example

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1	✔		
10	✔		
11			

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1	✔
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0	✔	100
1	✔	
10	✔	
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Example

0	✓	100	✓
1	✓		
10	✓		
11	✗		

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0	✔	100	✔
1	✔	101	
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0	✔	100	✔
1	✔	101	✔
10	✔	111	✖
11	✖	1000	

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10	✔	111	✖
11	✖	1000	✔

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Example

0	✓	100	✓	1001	✓
1	✓	101	✓	1011	✓
10	✓	111	✗	1010	✗
11	✗	1000	✓	1100	✗

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❗ each (except 0, 1) starts with 10

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! each (except 0, 1) starts with 10

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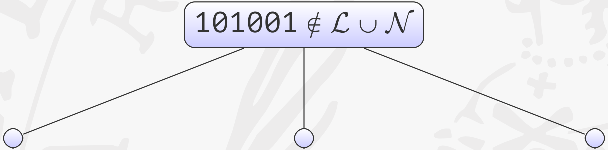
! each is bigger than its **Nyldon** suffixes (Charlier et al. (2019))

The Problem with Nyldon words

Example



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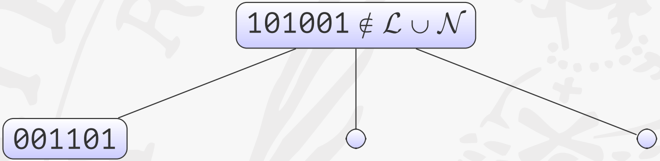

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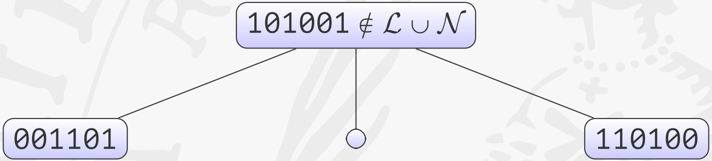

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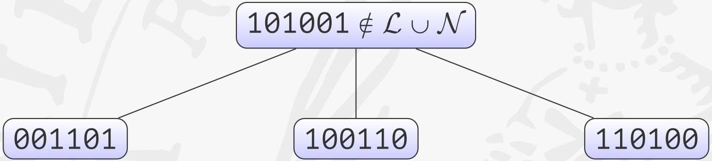

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for Nyldon words

Let $w = ps$ and s longest proper Nyldon suffix.

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$w \in \mathcal{N} \iff p \in \mathcal{N}$ *and* $p \rhd s$.

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Example

$\mathtt{10110}, \mathtt{10} \in \mathcal{N}$

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Example

$\mathtt{10110}, \mathtt{10} \in \mathcal{N} \quad \rightarrow \quad \mathtt{10110} \cdot \mathtt{10} \in \mathcal{N}$

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$\mathtt{10110}, \mathtt{10} \in \mathcal{N}$ → $\mathtt{10110} \cdot \mathtt{10} \in \mathcal{N}$ $\mathtt{10110}, \mathtt{1} \in \mathcal{N}$

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- $w = n_1 n_2 \cdots n_k$ where $n_i \in \mathcal{N}$ for $i \in [k-1]$ and $n_k \in \mathcal{N}_\omega$ with $n_i \trianglelefteq n_{i+1}$

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Example

$\texttt{10}^\omega$

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Example

$\mathtt{10}^\omega, \mathtt{101}^\omega \in \mathcal{N}_\omega$

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$\mathtt{10}^\omega, \mathtt{101}^\omega \in \mathcal{N}_\omega$ $(\mathtt{10})^\omega$

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Example

$10^\omega, 101^\omega \in \mathcal{N}_\omega$ $\qquad$ $(10)^\omega, 1010^\omega \notin \mathcal{N}_\omega$

Some results on infinite Nyldon words

Inf. Nyl.
Fact.

Some results on infinite Nyldon words

Inf. Nyl. Fact.

every $w \in \Sigma^\omega$ has unique infinite Nyldon factorisation

$(10)^\omega \notin \mathcal{N}_\omega$

$\downarrow$

$(10, 10, \ldots)$

$1010^\omega \notin \mathcal{N}_\omega$

$\downarrow$

$(10, 10^\omega)$

Some results on infinite Nyldon words

Inf. Nyl. Fact.

every $w \in \Sigma^\omega$ has unique infinite Nyldon factorisation

Prefix

Some results on infinite Nyldon words

Inf. Nyl. Fact.

every $w \in \Sigma^\omega$ has unique infinite Nyldon factorisation

Prefix

for $w = ps \in \mathcal{N}_\omega$ with $s \in \mathcal{N}_\omega$: $p \in \mathcal{N}$

$\mathtt{10110}^\omega \in \mathcal{N}_\omega$

$\downarrow$

$\mathtt{101} \in \mathcal{N}$

Some results on infinite Nyldon words

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for $w = ps \in \mathcal{N}_\omega$ with $s \in \mathcal{N}_\omega$: $p \in \mathcal{N}$

Suffix

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↓

$101 \in \mathcal{N}$

Some results on infinite Nyldon words

Inf. Nyl. Fact.	every $w \in \Sigma^\omega$ has unique infinite Nyldon factorisation
Prefix	for $w = ps \in \mathcal{N}_\omega$ with $s \in \mathcal{N}_\omega$: $p \in \mathcal{N}$
Suffix	for $w \in \mathcal{N}_\omega$ and $s \in \mathcal{N}_\omega$ with $s <_s w$: $s \lhd w$

$10110^\omega \in \mathcal{N}_\omega$

↓

$101 \in \mathcal{N}$

$10^\omega \lhd 10110^\omega$

Some results on infinite Nyldon words

Inf. Nyl. Fact. — every $w \in \Sigma^\omega$ has unique infinite Nyldon factorisation

Prefix — for $w = ps \in \mathcal{N}_\omega$ with $s \in \mathcal{N}_\omega$: $p \in \mathcal{N}$

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Standard Fact.

$\mathtt{10110}^\omega \in \mathcal{N}_\omega$

↓

$\mathtt{101} \in \mathcal{N}$

$\mathtt{10}^\omega \lhd \mathtt{10110}^\omega$

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Standard Fact.	either no inf. Nyldon suffix or similar to the finite case

$\mathtt{10110}^\omega \in \mathcal{N}_\omega$

$\downarrow$

$\mathtt{101} \in \mathcal{N}$

$\mathtt{10}^\omega \lhd \mathtt{10110}^\omega$

$\mathtt{101}^\omega$ has no infinite Nyldon suffix

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$\mathtt{10110}^\omega \in \mathcal{N}_\omega$

$\downarrow$

$\mathtt{101} \in \mathcal{N}$

$\mathtt{10}^\omega \lhd \mathtt{10110}^\omega$

$\mathtt{101}^\omega$ has no infinite Nyldon suffix

Further Research

- Infinite Nyldon words
  - Thue-Morse word
  - Fibonacci word
  - …

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- Generalised Nyldon words
  - infinite words

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Thank you for your attention!