Rational trace relations[1]

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Technische Universität Ilmenau

[1] Submitted to FSTTCS 2023

A class of rational trace relations closed under composition[1]

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   L regular $\Rightarrow$ $L^R = \{v \mid \exists u \in L\colon (u,v) \in R\}$ and
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   (similarly for many other language classes)

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   $\Longrightarrow$ reachability of regular sets of configurations decidable

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- R_2 left-closed $\implies [R_1 \circ R_2] = [R_1] \circ [R_2]$.
- R_1 and R_2 left-closed $\implies R_1 \circ R_2$ left-closed.

Further properties

Let $\mathcal{K}, \mathcal{L} \subseteq \mathbb{M}$ non-empty, $\mathcal{R} \subseteq \mathbb{M} \times \mathbb{M}$.

1. $\mathcal{K} \times \mathcal{L}$ lc-rational iff
   - $\mathcal{K}$ recognizable (i.e., $\{u \in \Sigma^* \mid [u] \in \mathcal{K}\}$ regular) and
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- forwards reachability preserves rationality:
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