Automata Giving Small Certificates for Large Solutions

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based on joint work with
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The Connoisseur of Number Sequences

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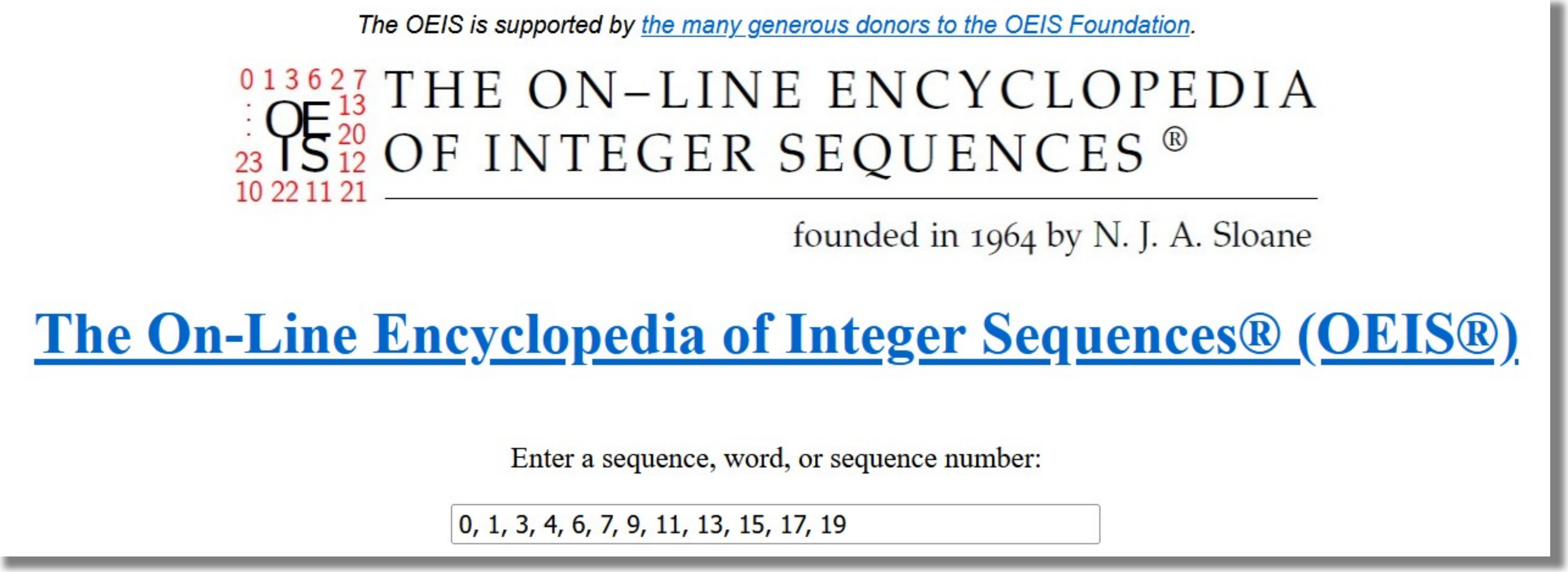
Neil Sloane (*1939)

OEIS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Numberphile

Problems with Powers of Two Problem

Given a set of integers S, denote by $b(S)$ the number of powers of two that can be obtained as the sum of two elements of S.

Examples:

- $S = \{1, 3\},\ b(S) = 1$
- $S = \{-1, 3, 5\},\ b(S) = 3$
- $S = \{-3, -1, 3, 5\},\ b(S) = 4$

Problems with Powers of Two Problem

Denote by $a(n)$ the largest value of $b(S)$ that can be achieved for a set S with n elements.

n	1	2	3	4	5	6	7	8	9	10	11	12
$a(n)$	0	1	3	4	6	7	9	11	13	15	17	19

Largest known value: $a(18) = 34$

Upper bound: $$a(n) \leq \frac{n}{4} \cdot \sqrt{4n-3} + 1$$

OEIS A352178

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Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A352178 Let S = {t_1, t_2, ..., t_n} be a set of n distinct integers and consider the sums t_i + t_j (i<j); a(n) is the maximum number of such sums that are powers of 2, over all choices for S. 4

0, 1, 3, 4, 6, 7, 9, 11, 13, 15, 17, 19, 21, 24, 26, 29, 31, 34 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,3

COMMENTS

Given distinct integers t_1, ..., t_n, form a graph G with n vertices labeled by the t_i, and with an edge from t_i to t_j, labeled t_i + t_j, whenever t_i + t_j is a power of 2.

See the Pratt link for the best lower bounds known, and examples of sets achieving these bounds, for 1 <= n <= 100. - N. J. A. Sloane, Sep 26 2022

The following remarkable theorem is due to M. S. Smith (email of Mar 06 2022).

Theorem: G contains no 4-cycles.

Proof. Suppose the contrary, and assume the vertices t_1, t_2, t_3, t_4 form a 4-cycle, with edges labeled b_1 = t_1+t_2, b_2 = t_2+t_3, b_3 = t_3+t_4, b_4 = t_4+t_1. The b_i are powers of 2.

Since the t_i are distinct, b_1 != b_4, b_2 != b_1, b_3 != b_2, and b_4 != b_3.

We also have

Open Problems and Challenges

- How does A352178 continue?

  Nobody knows! (but some lower and upper bounds are known)

- Is it possible to continue A352178, at least in theory?

  Nobody knew, iterating over all sets with n integers is not possible...

A Logicians View on the Problem

To determine whether $a(n) \geq k$:

- Find integers $z_1, z_2, \ldots, z_n$

  $\exists z_1, z_2, \ldots, z_n$

- For every pair $i < j$ an indicator variable $x_{i,j} \in \{0,1\}$ assigning 1 to $x_{i,j}$ exactly when $z_i + z_j$ is a power of 2

  $$\exists x_{1,2} x_{1,3} \ldots x_{n-1,n}\, P_2(z_1 + z_2) \rightarrow x_{1,2} = 1 \wedge$$
  $$\neg P_2(z_1 + z_2) \rightarrow x_{1,2} = 0 \wedge \cdots$$

- The sum of all indicator variables is at least k

  $x_{1,2} + x_{1,3} + \cdots + x_{n-1,n} \geq k$

Büchi Arithmetic

Logical formula obtained is statement in Büchi arithmetic, which is an automatic structure:

- Numbers are just sequences of digits
- Can define DFA for basic relations
- Use closure properties of regular languages under boolean operations, homomorphisms and inverse homomorphisms to decide logical theory

J.R. Büchi
(1924 – 1984)

Véronique
Bruyère

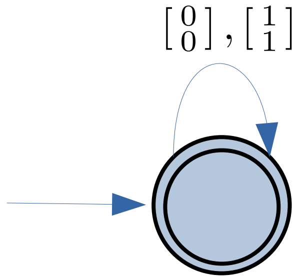
Presburger Arithmetic

First-order theory of $(\mathbb{N}, 0, 1, +, =)$

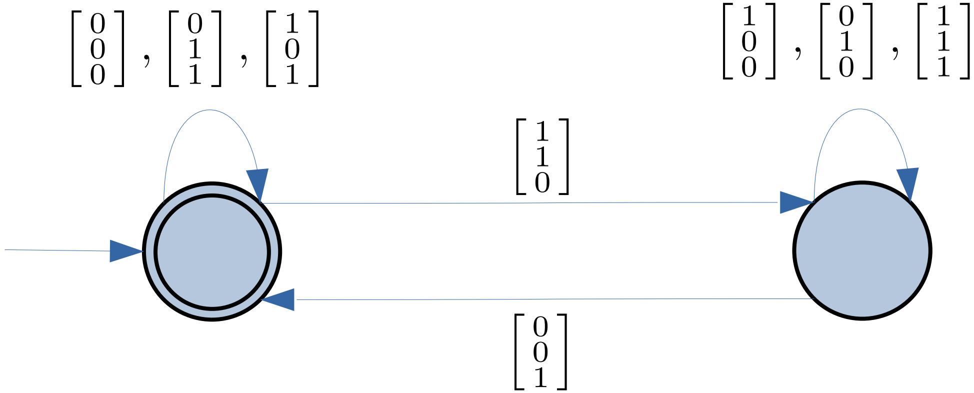
Represent $\boldsymbol{x} \in \mathbb{N}^d$ as strings over the alphabet

$$\Sigma_d = \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

Gadget for $x = y$:



Gadget for $x + y = z$:

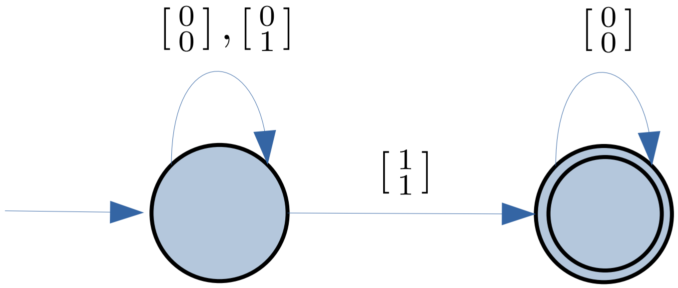


Büchi Arithmetic

First-order theory of $(\mathbb{N}, 0, 1, +, V_p, =)$ for fixed $p > 1$:

$V_p(x, y) \Leftrightarrow$ x is the largest power of p dividing y without remainder

Gadget for $V_2(x, y)$:



<u>Theorem</u> (Büchi, 1960, Bruyere 1985; H., Różycki , 2021)
Sets definable in Büchi arithmetic coincide with regular languages. Büchi arithmetic is not model-complete.

Dealing with Negative Numbers

Sloane's problem requires looking for integer solutions:

- Encode numbers in base -2:

$$23 = 1 \cdot (-2)^0 + 1 \cdot (-2)^1 + 0 \cdot (-2)^2 + 1 \cdot (-2)^3 + 0 \cdot (-2)^4 + 1 \cdot (-2)^5 + 1 \cdot (-2)^6$$

- DFA for addition becomes a bit more complicated:

A Partial Answer to the Power-of-Two Problem

The constructed NFA shrink the search space:

- There is a constant c such that to check whether $a(n) \geq k$, it suffices to "only" consider sets with integers in the interval

$$\{-2^{2^{c \cdot n}}, \ldots, 2^{2^{c \cdot n}}\}$$

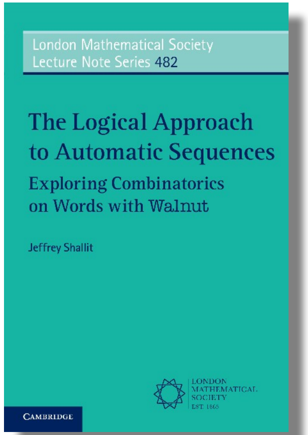
The NFA become huge – $a(3) \geq 3$

Practical Aspects

The vanilla logical approach not truly advantageous in practice:

- Walnut quickly runs out of resources
- SMT-solvers do not support power-of-two assertions, but ad-hoc approach finds in reasonable time $a(18) = 34$
- Still $a(19)$ hits the wall

Jeffrey Shallit
(*1957)



Solutions May Have Super-Polynomial Bit Length

Theorem (Matthews, 1982)
There are infinitely many primes p such that the multiplicative order of two is at least $\sqrt{p}$.

Multiples of multiplicative order of two definable in Büchi arithmetic:

$$\Phi_p(x) \equiv x > 1 \wedge P_2(x) \wedge \exists y\, x - p \cdot y = 1$$

Theorem (Guepin, H., Worrell, 2019)
Existential Büchi arithmetic is NP-complete.

An Alternative DFA construction

DFA accepting solutions of system of equations $\boldsymbol{a} \cdot \boldsymbol{x} = c$

$M = (Q, \{0,1\}^d, \delta, q_0, F)$:

- $Q = \mathbb{Z} \cup \{\perp\}$
- $q_0 = 0$
- $\delta(z, \boldsymbol{u}) = 2z + \boldsymbol{a} \cdot \boldsymbol{u}$ for all $z \in \mathbb{Z}$
- $\delta(\perp, \boldsymbol{u}) = \perp$ for all $\boldsymbol{u} \in \{0,1\}^d$
- $F = \{c\}$

After reading $\boldsymbol{u}_{n-1} \cdots \boldsymbol{u}_0 \in \{0,1\}^d$, automaton is in state

$$\sum_{i=0}^{n-1} \boldsymbol{a} \cdot \boldsymbol{u}_i$$

Certificates Witnessing Existence of Solutions

Given configurations $\boldsymbol{q}, \boldsymbol{r} \in \mathbb{Z}^m$ of DFA for $A \cdot \boldsymbol{x} = \boldsymbol{c}$ reading solutions in base p, have

$\boldsymbol{q}$ reaches $\boldsymbol{r}$

$$\iff$$

there exists $k \in \mathbb{N}$ and $\boldsymbol{u} \in \{0, \ldots, p^k - 1\}^d$ such that

$$\boldsymbol{r} = p^k \cdot \boldsymbol{q} + A \cdot \boldsymbol{u}$$

To decide existential Büchi arithmetic in NP, guess polynomially many configurations and check reachability

p-Universality

- p-universality: Given existential formula of Büchi arithmetic, is it satisfiable in every (prime) base p?
- Number of states of DFA for $A \cdot \boldsymbol{x} = \boldsymbol{c}$ independent of base p

Theorem (H., Mansutti, 2021)
Deciding p-universality is coNEXP-complete.

Existential Presburger Arithmetic with Divisbility

p-universality cornerstone of decidability of existential Presburger arithmetic with divisibility:

- Atomic formulas:

$$(a_1 \cdot x_1 + \cdots + a_d \cdot x_d + a_0) \mid (b_1 \cdot x_1 + \cdots + b_d \cdot x_d + b_0)$$

- Generalizes systems of linear congruences

- Smallest solutions can be large: $x_n \geq 2^{2^n}$ for

$$\Phi_n \equiv x_n > 1 \wedge \bigwedge_{i=0}^{n-1} x_i > 1 \wedge (x_i \mid x_{i+1}) \wedge (x_i + 1 \mid x_{i+1})$$

Existential Presburger Arithmetic with Divisbility

Lipshitz (1978) showed certain local-to-global property:

- Every formula Φ is equi-satisfiable with some Ψ in *increasing form* such that Ψ has a solution in $\mathbb{Z}$ iff

  Ψ has a solution modulo every prime p, i.e., for every $f(\boldsymbol{x}) \mid g(\boldsymbol{x})$ in Ψ, find $\boldsymbol{x}_p$ such that
  $$f(\boldsymbol{x}_p) \neq 0, \ v_p(f(\boldsymbol{x}_p)) \leq v_p(g(\boldsymbol{x}_p))$$

- Allows to deduce NEXP upper bound

Complexity of Increasing Formulas

Theorem (Defossez, H., Mansutti, Perez, 2023)
Increasing formulas of existential Presburger arithmetic with divisibility are decidable in NP.

- Show that only a polynomial number of prime numbers are essential for local-to-global property
- Then perform polynomial number of queries to existential Büchi arithmetic

From Local to Global Solutions

Going from local to global solutions requires combining solutions modulo p via the Chinese remainder theorem

Theorem (Defossez, H., Mansutti, Perez, 2023)

If a system of congruences and non-congruences

$$x \equiv b_m \pmod{m} \qquad m \in M$$

$$x \not\equiv c_{q,i} \pmod{q} \qquad q \in Q \subseteq \mathbb{P}, 1 \leq i \leq d$$

has a solution then it has a solution bounded by

$$\prod M \cdot ((d+1) \cdot \#Q)^{4(d+1)^2(3+\ln\ln(\#Q+1))}$$

Integer Programming with GCD constraints

With further technical developments, can show small-model property for generalisation of integer programming:

$$\begin{array}{lll} \text{minimize} & \boldsymbol{c} \cdot \boldsymbol{x} & \\ \text{subject to} & A \cdot \boldsymbol{x} \leq \boldsymbol{b} & \\ & \gcd(f_i(\boldsymbol{x}), g_i(\boldsymbol{x})) \sim_i d_i, & 1 \leq i \leq k \end{array}$$

Theorem (Defossez, H., Mansutti, Perez, 2023)
If an instance of IP-GCD is feasible then it has a solution (and an optimal solution, if one exists) of polynomial bit length. Hence, IP-GCD feasibility is NP-complete.

Semenov arithmetic

First-order theory of the structure $(\mathbb{N}, 0, 1, +, 2^x)$

- Shown decidable by Semenov in 1984
- Model-complete and admits quantifier eliminiation (Cherlin and Point, 1986)
- Non-elementary complexity in general:

$$p\colon (x, y) \mapsto 2^{2^x} + 2^{2y+1}$$

- Existential fragment decidable in NEXP (Benedikt et al. 2023), but also decidable if regular predicates allowed?

Semenov arithmetic is not automatic

<u>Theorem</u> (Khoussainov, Nerode, 1995)
If $f\colon \mathbb{U}^n \to \mathbb{U}$ is a function whose graph is a regular relation then there is a constant $C > 0$ such that for all $u_1, \ldots, u_n \in \mathbb{U}$:

$$|f(u_1, \ldots, u_n)| \leq \max(|u_1|, \ldots, |u_n|) + C$$

- No such constant exists for $f\colon x \mapsto 2^x$
- Quantifier-elimination incompatible with regular predicates, since theory undecidable if V_2 included

Power functions

Suppose we have $x = 2^y$ then:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} \cdots \\ \cdots \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Length of string of tailing zeros of x equals value of y
- Keep two counters when reading tuple of numbers:
  - First counter tracks number of tailing zeros of x
  - Second counter computes value of y
- Accept if both counters have the same value

Affine Vector Addition Systems with States (VASS)

- Finite-state automata with finite number of counters taking values from $\mathbb{N}$
- Transitions update counters by affine functions

$$f : x \mapsto a \cdot x + b, \;\; a, b \in \mathbb{Z}$$

- Languages closed under union and intersection

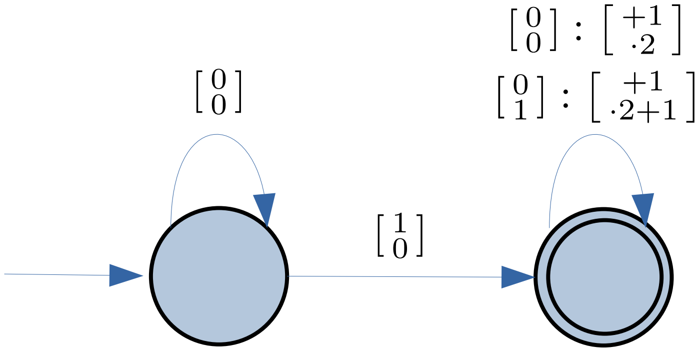
Theorem (Finkel, Göller, H., 2013; Reichert, 2015; Jaax, Kiefer, 2020)
Reachability in affine VASS with a single counter is PSpace-complete and undecidable for two counters.

Affine VASS for power functions

Suppose we have $x = 2^y$ then:

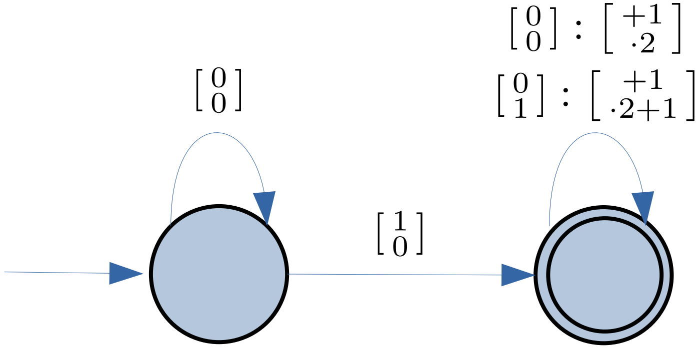
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{matrix} \cdots \\ \cdots \end{matrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Affine 2-VASS accepting such strings



Accept if in accepting state and both counters have same value

Restricted affine VASS



- Counters can be grouped into pairs
- Once first counter of pair has value 1, it gets incremented and only incremented at every transition
- Languages of restricted affine VASS closed under union and intersection

A counter elimination procedure

For deciding emptiness with arbitrary number of pairs:

- Guess ordering in which counters become non-zero
- Successively eliminate counters until finite-state automaton is obtained with language equi-nonempty
- Number of control states squares in every iteration

Theorem (Draghici, H., Manea, 2023)
Deciding language emptiness of restricted affine VASS is in EXPSpace.

A Decision Procedure

To decide existential $(\mathbb{N}, 0, 1, +, 2^x, (R_k)_{k>0})$

- Formula given as positive Boolean combination of
  - linear equations $a_1 \cdot x_1 + \cdots + a_n \cdot x_n = b$
  - $R_i(x_1, \ldots, x_{d_i})$
  - applications of powering function $x = 2^y$

- Yields equi-nonempty exponential-size restricted affine VASS

Theorem (Draghici, H., Manea, 2023)
Existential $(\mathbb{N}, 0, 1, +, 2^x, (R_k)_{k>0})$ is in EXPSpace.

String Constraints

Two-sorted logic with (subset of) consisting of:

- Systems of equations of the form:

$$x \cdot y = z, \ \ x, y, z \in \{0,1\}^*$$

- Length function $\ell \colon \{0,1\}^* \to \mathbb{N}$:

$$\ell \colon b_0 \cdots b_k \mapsto k + 1$$

- Number-to-string conversions $\mathrm{nr2str} \colon \mathbb{N} \to \{0,1\}^*$:

$$\mathrm{nr2str} \colon n \mapsto \left\{ b_0 \cdots b_k : n = \sum_{i=0}^{k} 2^i \cdot b_i \right\}$$

- Regular membership: $R(x)$
- Presburger constraints: $\Phi(u_1, \ldots, u_n)$

Word equations

Theorem (Makanin, 1977; Jez 2017)
Simple word equations are decidable in non-deterministic linear space.

Theorem (Berzish et al., 2021)
String constraints with length constraints, number-to-string functions and regular language membership are undecidable.

A decidable fragment

Theorem (Draghici, H., Manea, 2023)
The existential theory of string constraints with length constraints

- number-to-string constraints, and
- regular language membership

polynomial-time reduces to $(\mathbb{N}, 0, 1, +, 2^x, (R_k)_{k>0})$ and is hence decidable in EXPSpace.

Idea: Map $s \in \{0,1\}^*$ to $1 \cdot s$