Properties of polyregular functions

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GI Theorietag 2023

Kaiserslautern

Regular functions

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Regular languages are the ones "expressible" via

- → DFA
- → NFA
- → 2-way FA
- → regular expressions
- → MSO

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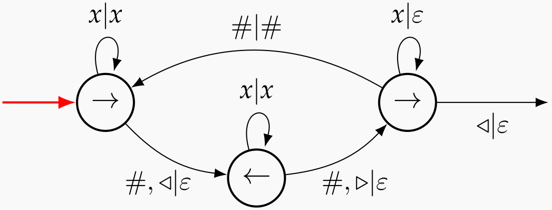
Let's generalise from languages $L \subseteq \Sigma^*$ to functions $f : \Sigma^* \to \Gamma^*$.
To this end, we consider transducers, automata with output.

→ Tito's transducer simulation ←

Two-way transducers (mentioned in [Shepherdson 1958]!)

Example: $w_1\# \dots \# w_n \longmapsto w_1 \cdot \texttt{reverse}(w_1)\# \dots \# w_n \cdot \texttt{reverse}(w_n)$



$(x \in \{a, b, c\})$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output:

Two-way transducers (mentioned in [Shepherdson 1958]!)

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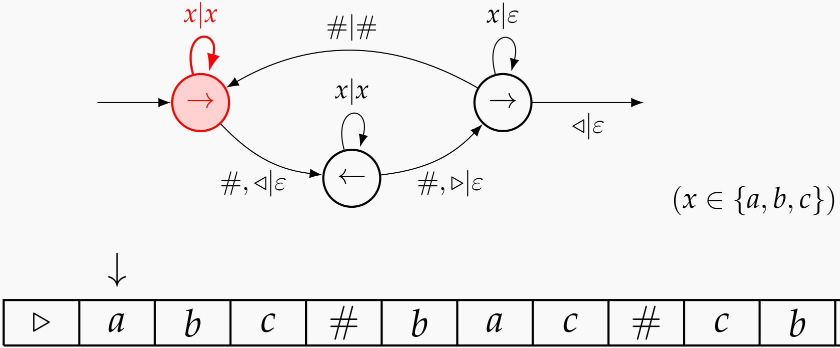

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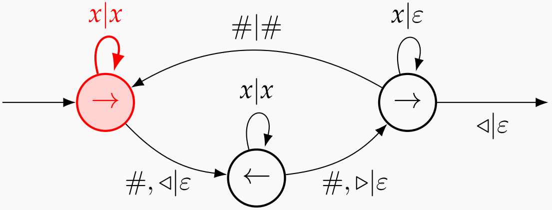

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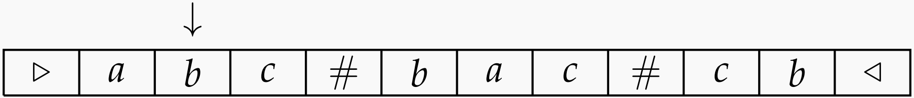

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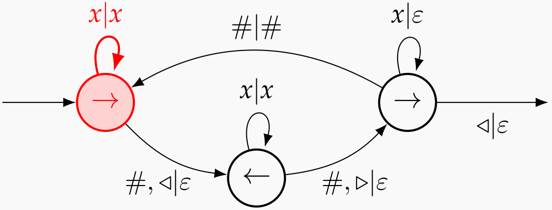

$(x \in \{a, b, c\})$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: ab

Two-way transducers (mentioned in [Shepherdson 1958]!)

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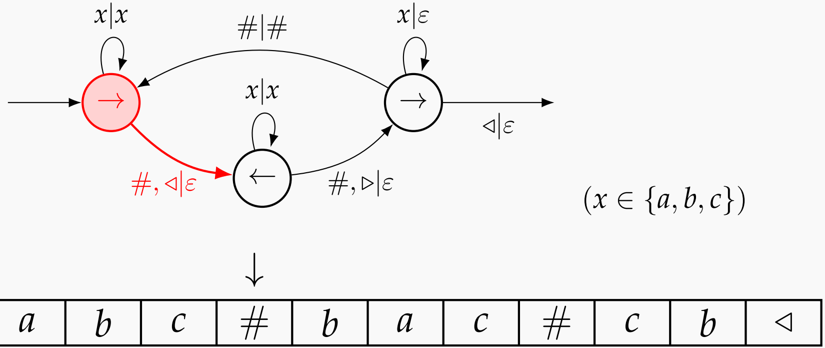

Output: abc

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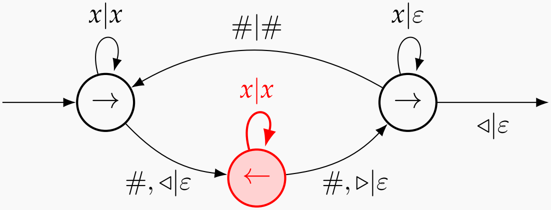

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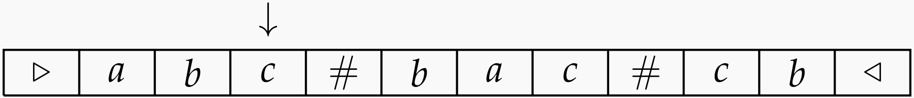

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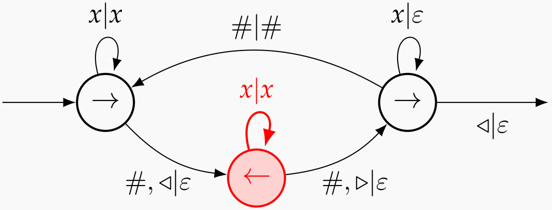

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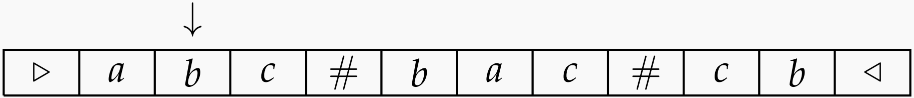

Output: $abcc$

Two-way transducers (mentioned in [Shepherdson 1958]!)

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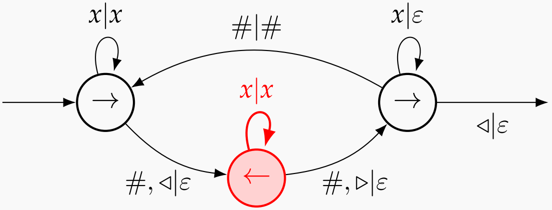

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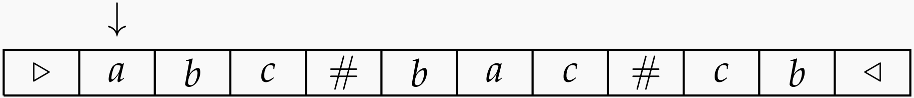

Output: *abccb*

Two-way transducers (mentioned in [Shepherdson 1958]!)

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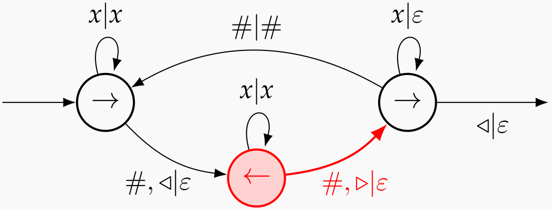

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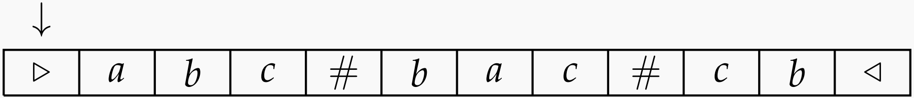

Output: *abccba*

Two-way transducers (mentioned in [Shepherdson 1958]!)

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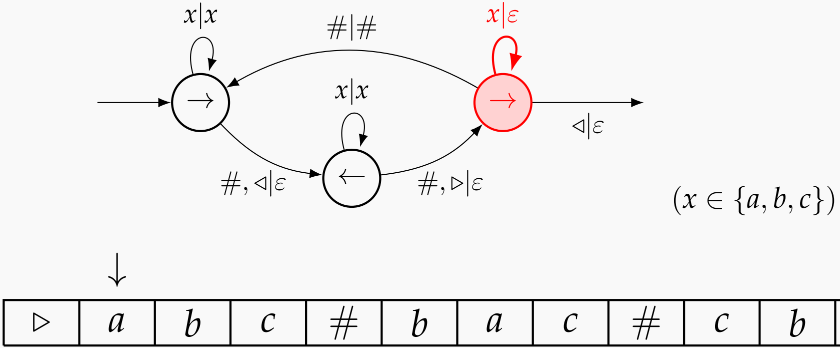

Output: *abccba*

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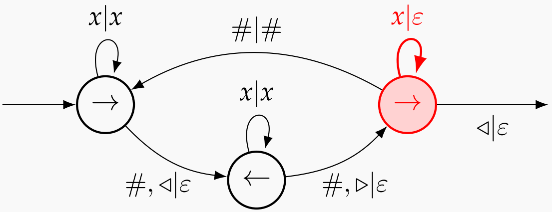

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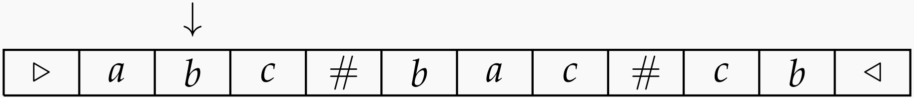

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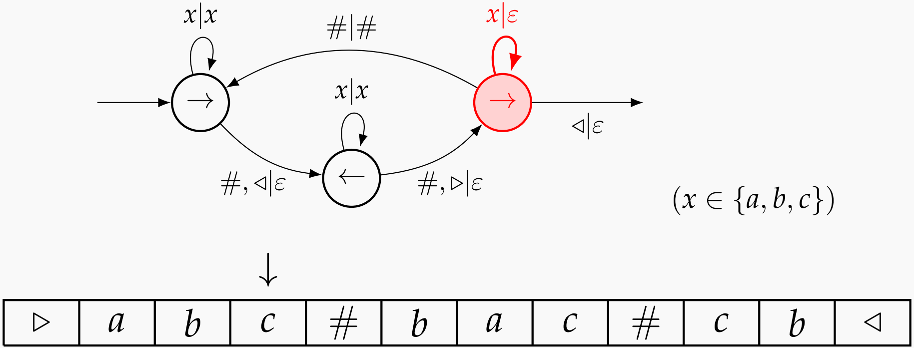

$(x \in \{a, b, c\})$

Output: *abccba*

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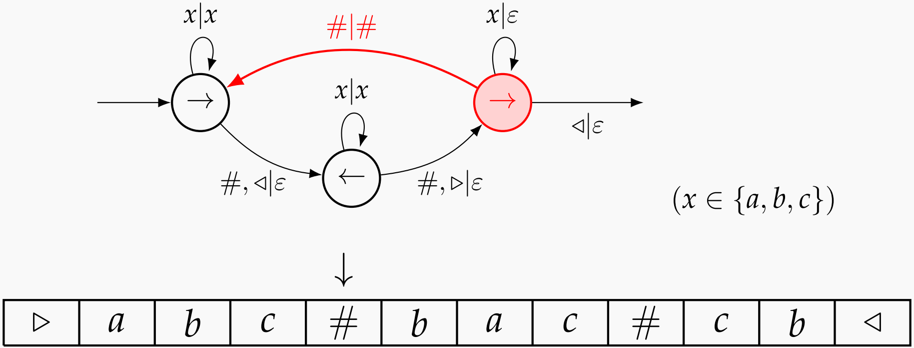

$(x \in \{a, b, c\})$

Output: *abccba*

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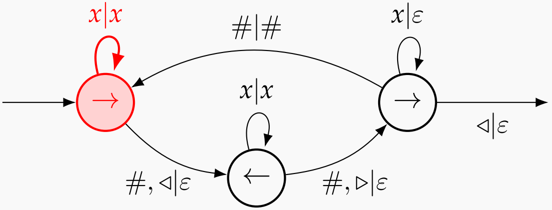

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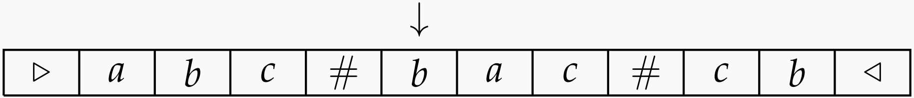
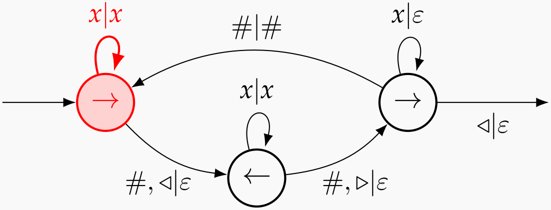


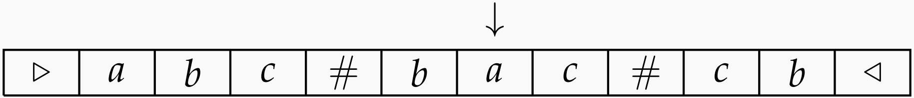
Output: $abccba\#$

Two-way transducers (mentioned in [Shepherdson 1958]!)

Example: $w_1\# \dots \# w_n \longmapsto w_1 \cdot \texttt{reverse}(w_1)\# \dots \# w_n \cdot \texttt{reverse}(w_n)$



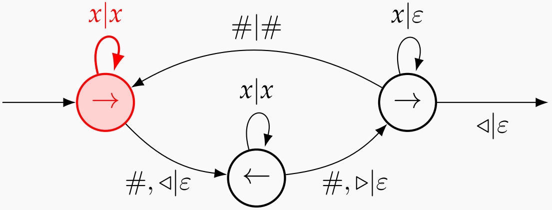
$(x \in \{a, b, c\})$



Output: $abccba\#b$

Two-way transducers (mentioned in [Shepherdson 1958]!)

Example: $w_1\# \ldots \# w_n \longmapsto w_1 \cdot \mathtt{reverse}(w_1) \# \ldots \# w_n \cdot \mathtt{reverse}(w_n)$



$(x \in \{a, b, c\})$

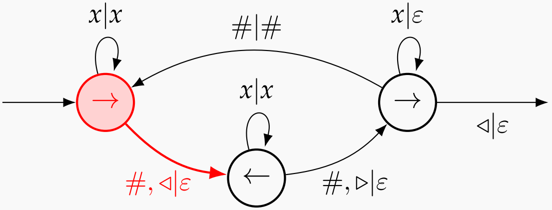
$\downarrow$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $abccba\#ba$

Two-way transducers (mentioned in [Shepherdson 1958]!)

Example: $w_1\# \dots \# w_n \longmapsto w_1 \cdot \texttt{reverse}(w_1)\# \dots \# w_n \cdot \texttt{reverse}(w_n)$



$(x \in \{a, b, c\})$

$\rhd$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\lhd$

Output: $abccba\#bac$

Two-way transducers (mentioned in [Shepherdson 1958]!)

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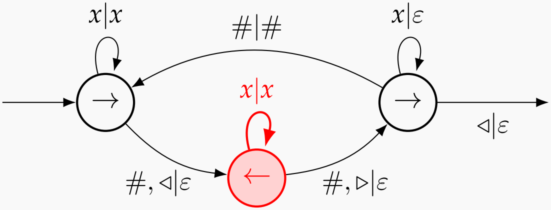

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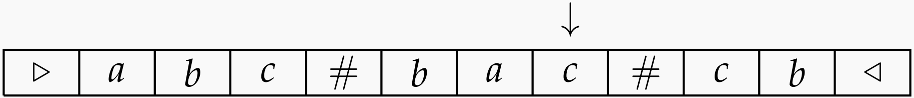

Output: $abccba\#bac$

Two-way transducers (mentioned in [Shepherdson 1958]!)

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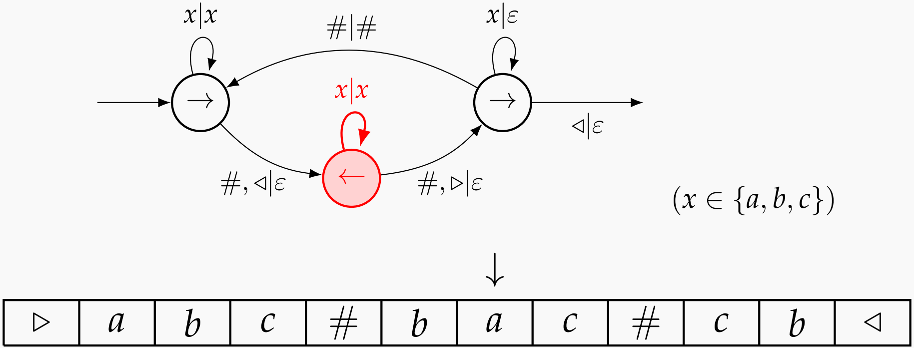

Output: $abccba\#bacc$

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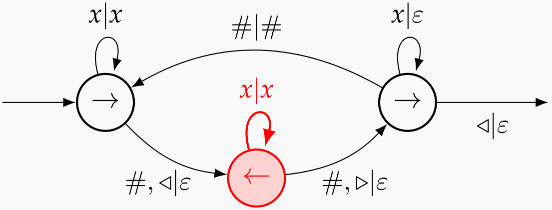

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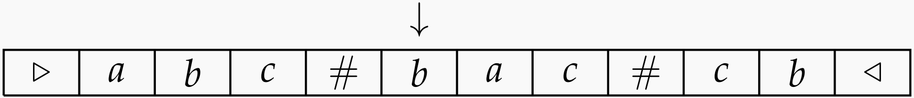

Output: $abccba\#bacca$

Two-way transducers (mentioned in [Shepherdson 1958]!)

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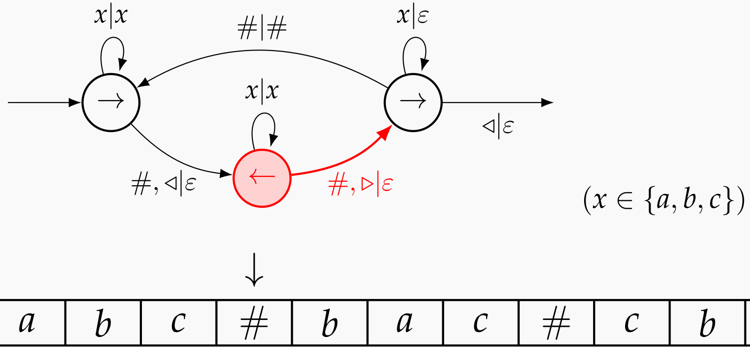

Output: $abccba\#baccab$

Two-way transducers (mentioned in [Shepherdson 1958]!)

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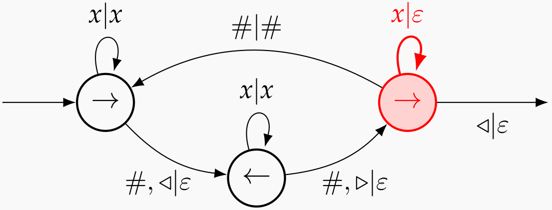

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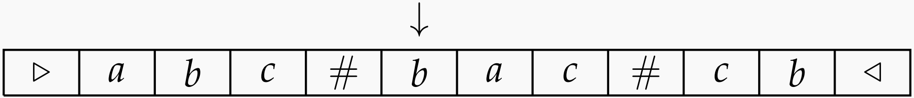

Output: $abccba\#baccab$

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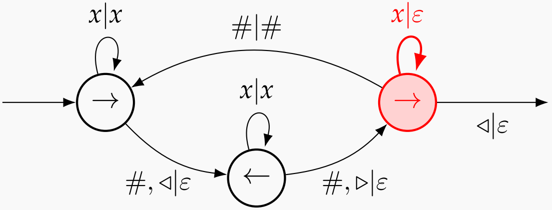

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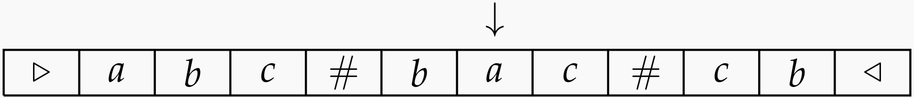

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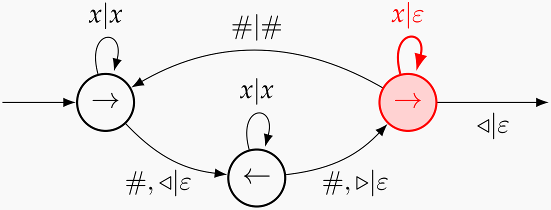

$(x \in \{a, b, c\})$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $abccba\#baccab$

Two-way transducers (mentioned in [Shepherdson 1958]!)

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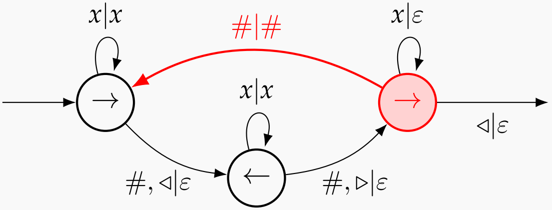

$(x \in \{a, b, c\})$

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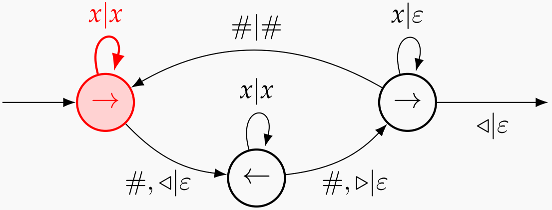


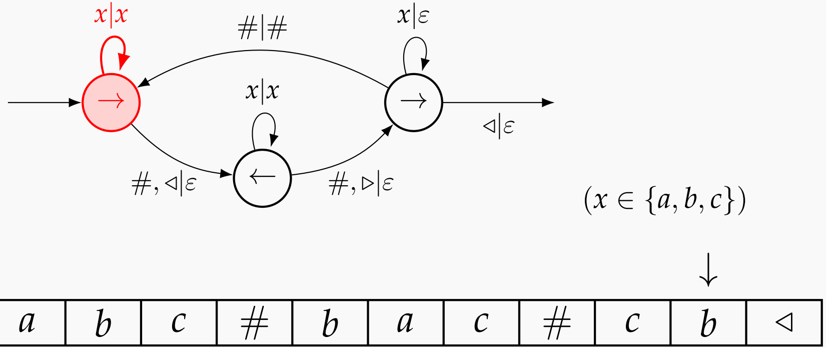
$(x \in \{a, b, c\})$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $abccba\#baccab\#$

Two-way transducers (mentioned in [Shepherdson 1958]!)

Example: $w_1\# \ldots \# w_n \longmapsto w_1 \cdot \texttt{reverse}(w_1)\# \ldots \# w_n \cdot \texttt{reverse}(w_n)$



Output: $abccba\#baccab\#c$

Two-way transducers (mentioned in [Shepherdson 1958]!)

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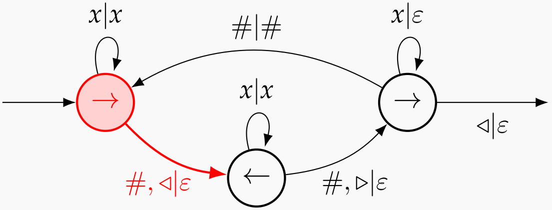

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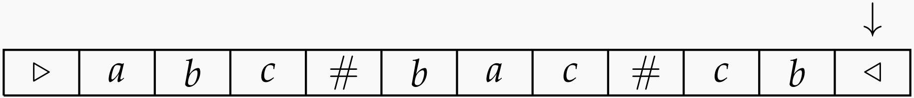

Output: $abccba\#baccab\#cb$

Two-way transducers (mentioned in [Shepherdson 1958]!)

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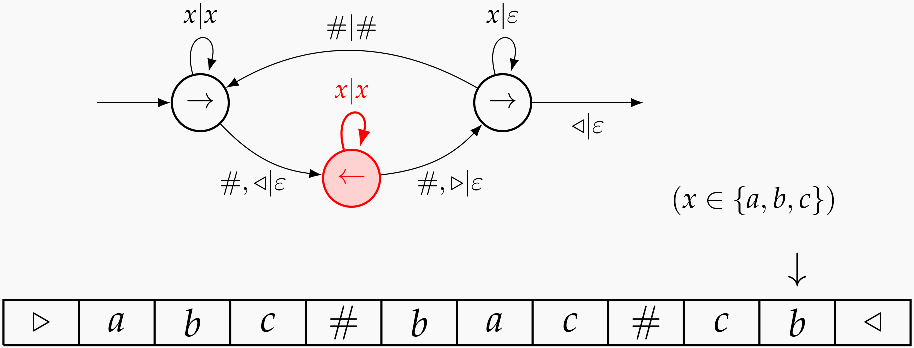

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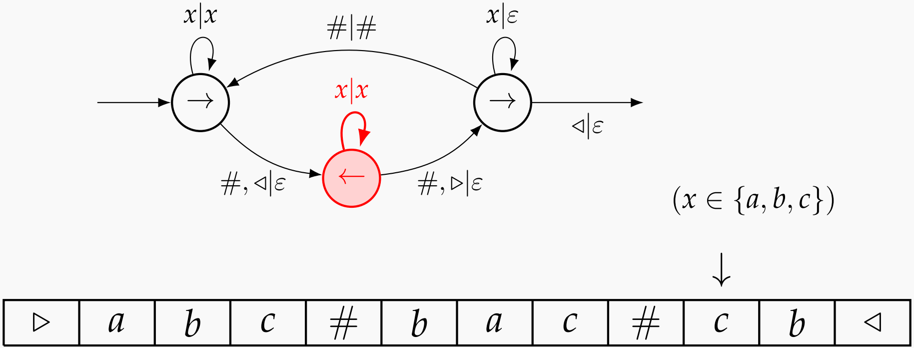

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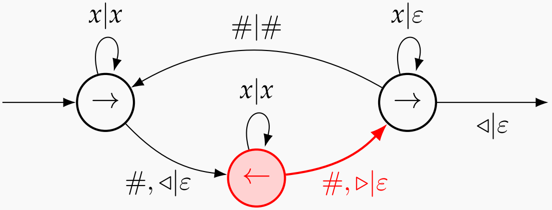
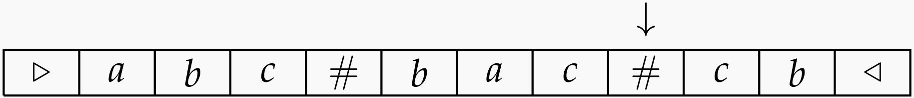


$(x \in \{a,b,c\})$



Output: $abccba\#baccab\#cbbc$

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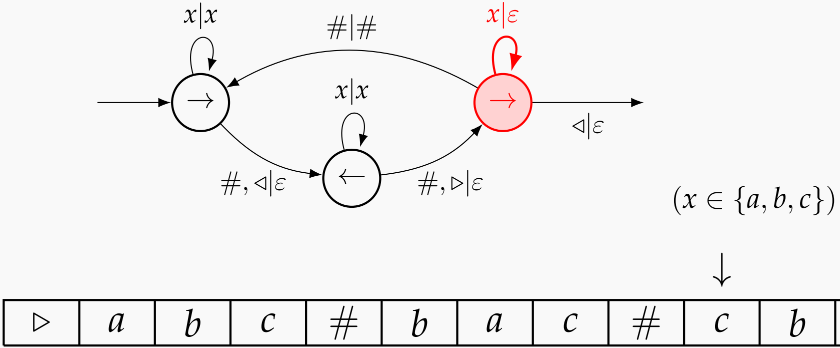

Output: $abccba\#baccab\#cbbc$

Two-way transducers (mentioned in [Shepherdson 1958]!)

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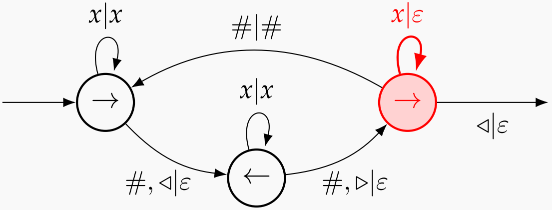
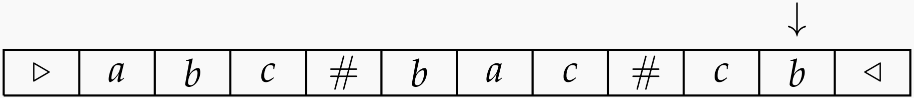


$(x \in \{a, b, c\})$



Output: $abccba\#baccab\#cbbc$

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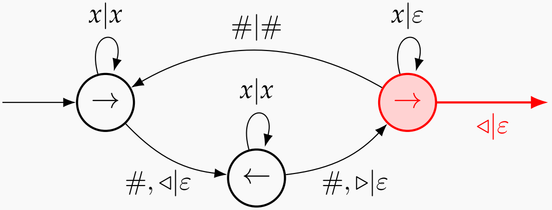

$(x \in \{a, b, c\})$

$\downarrow$

$\rhd$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\lhd$

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Regular functions

Regular languages are the ones "expressible" via
→ DFA
→ NFA
→ 2-way FA
→ regular expressions
→ MSO

Let's generalise from languages $L \subseteq \Sigma^*$ to functions $f : \Sigma^* \to \Gamma^*$.
To this end, we consider transducers, automata with output.

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Those form a well-understood class with nice properties:
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- preimages of regular languages are regular
- robust, many equivalent definitions, e.g. MSO transductions

From "regular" to "polyregular"

Regular functions = functions computed by deterministic 2-way transducers

For regular functions, the output length is always at most linear in the input length: $f(|w|) = \mathcal{O}(|w|)$ $\Rightarrow$ linear growth rate

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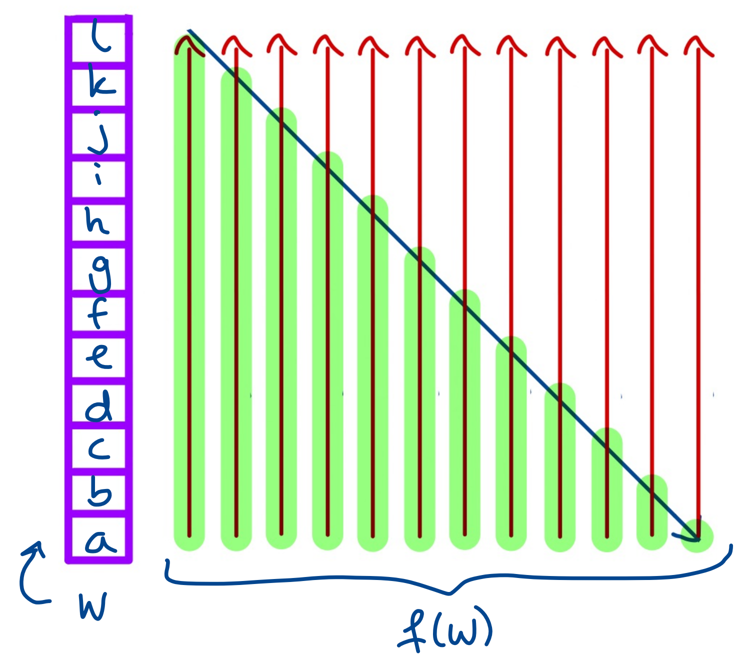
How can we modify the model to go <u>beyond</u> linear growth?

We equip the 2-way transducers with multiple reading heads, which can also serve as markers ("pebbles").

→ Tito's pebble transducer simulation ←



Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\texttt{innsq}: w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output:

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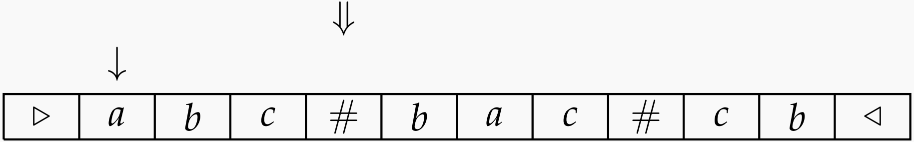

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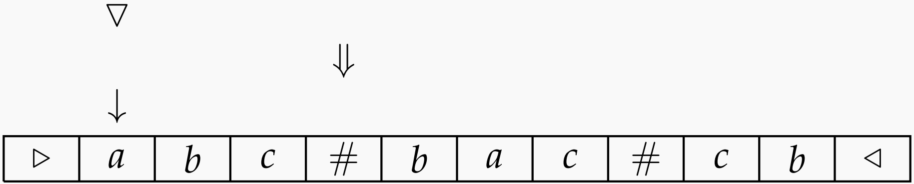

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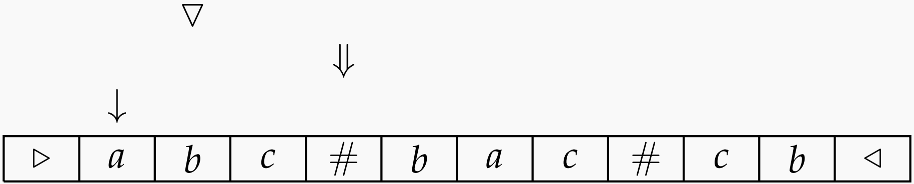

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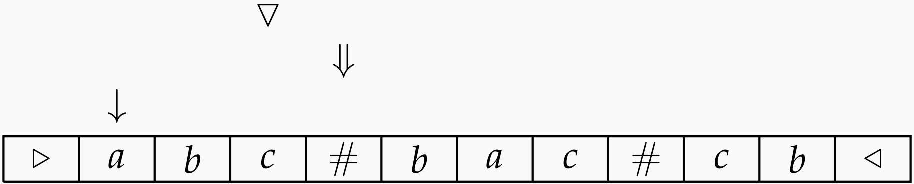

Output: $\quad ab$

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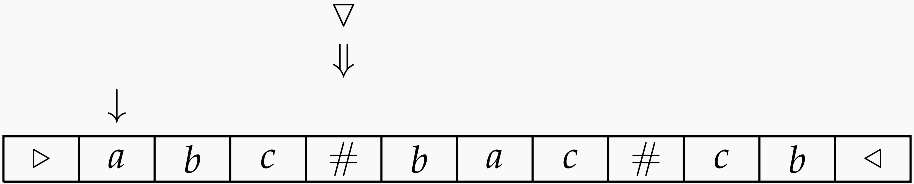

Output: *abc*

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Output: $\quad abc$

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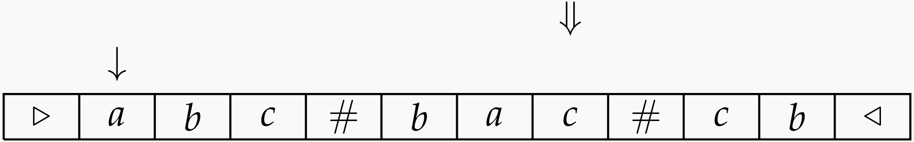

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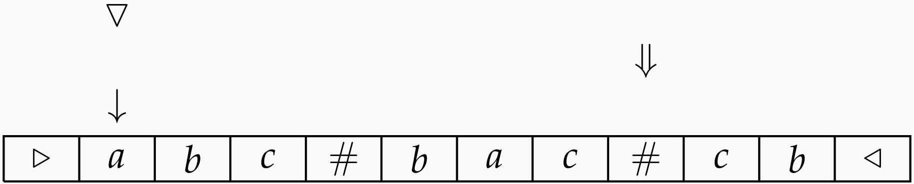

Output: $\quad abc$

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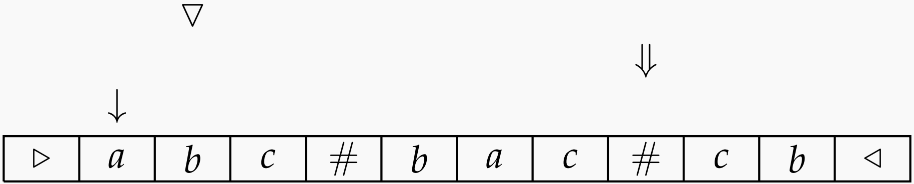

Output: *abca*

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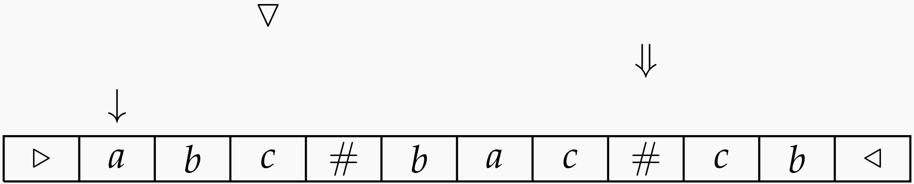

Output: *abcab*

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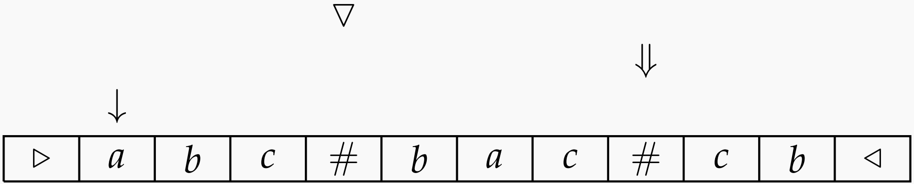

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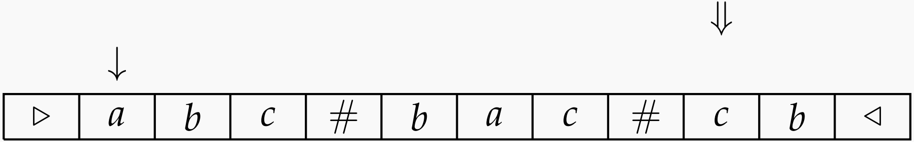

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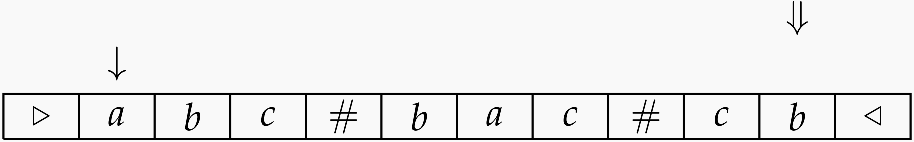

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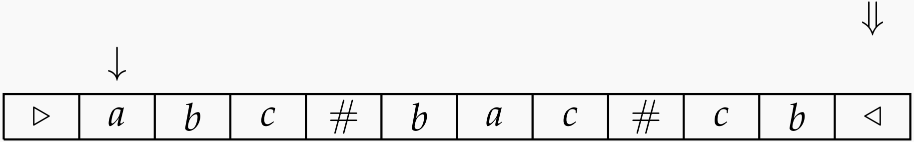

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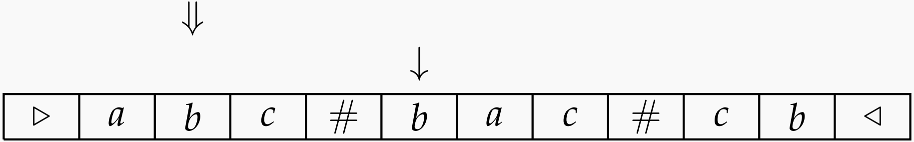

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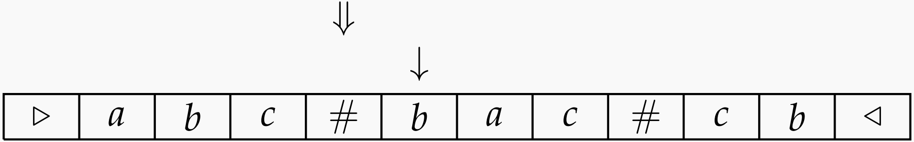

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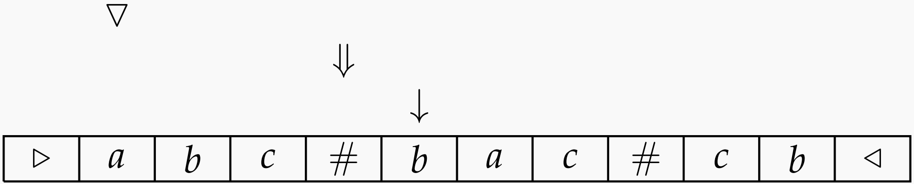

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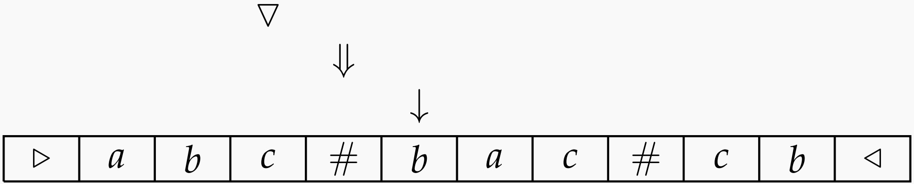

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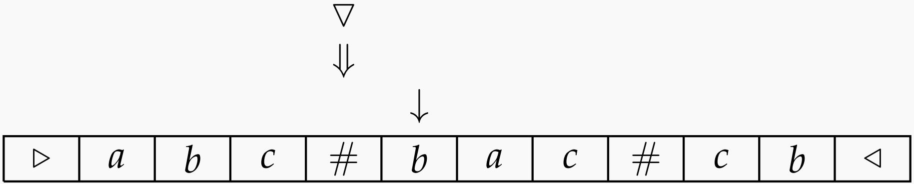

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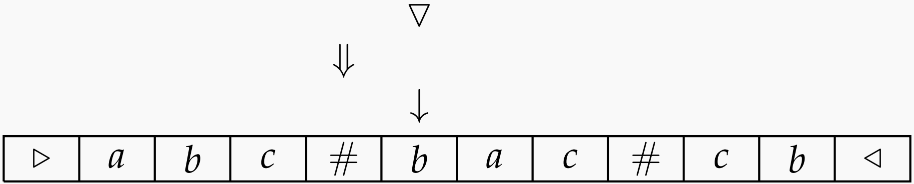
Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\mathtt{innsq}\colon w_0\#\ldots\#w_n \longmapsto (w_0)^n\#\ldots\#(w_n)^n$



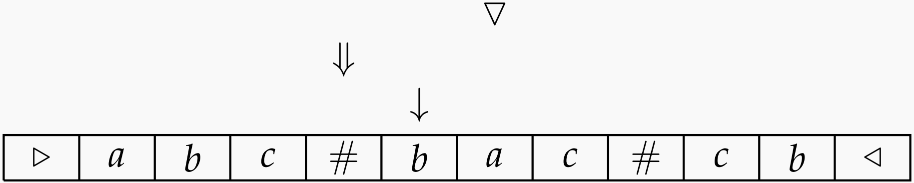
Output: *abcabc#*

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$



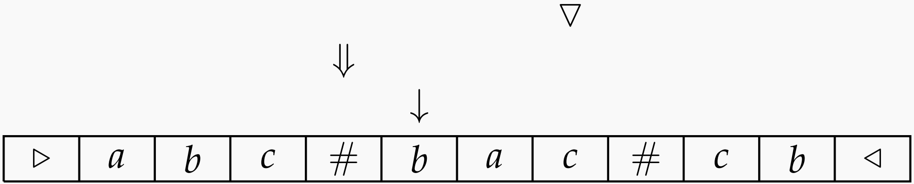
Output: *abcabc#b*

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \mathtt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$



Output: $\quad abcabc\#ba$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

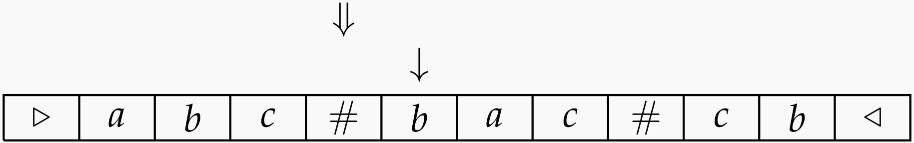
Output: $\quad abcabc\#bac$

Pebble transducers

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\texttt{innsq}\colon w_0\#\ldots\#w_n \longmapsto (w_0)^n\#\ldots\#(w_n)^n$



Output: *abcabc#bac*

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads (“pebbles”)

“Inner squaring” $\quad \texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$

					$\Downarrow$ $\downarrow$						
$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

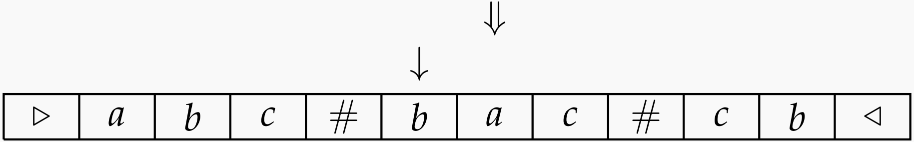
Output: $\quad abcabc\#bac$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\#\ldots\#w_n \longmapsto (w_0)^n\#\ldots\#(w_n)^n$



Output: $\quad abcabc\#bac$

Pebble transducers

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\texttt{innsq}\colon w_0\# \ldots \# w_n \longmapsto (w_0)^n \# \ldots \# (w_n)^n$

							$\Downarrow$				
					$\downarrow$						
$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

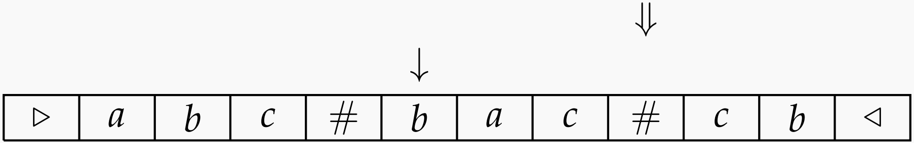
Output: $abcabc\#bac$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\#\ldots\#w_n \longmapsto (w_0)^n\#\ldots\#(w_n)^n$



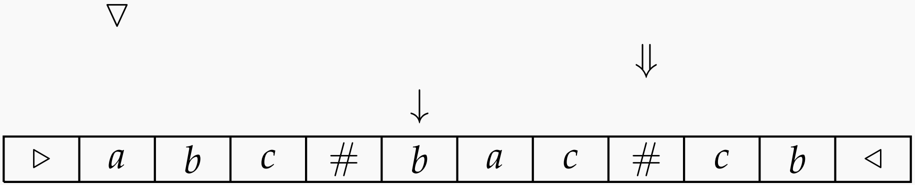
Output: $\quad abcabc\#bac$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \ldots \# w_n \longmapsto (w_0)^n \# \ldots \# (w_n)^n$



Output: *abcabc#bac*

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$

		$\triangledown$			$\downarrow$			$\Downarrow$			
$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $\quad abcabc\#bac$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \ldots \# w_n \longmapsto (w_0)^n \# \ldots \# (w_n)^n$

$\bigtriangledown$

$\Downarrow$

$\downarrow$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

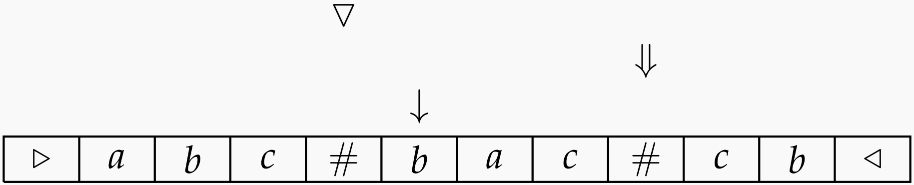
Output: $\quad abcabc\#bac$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$

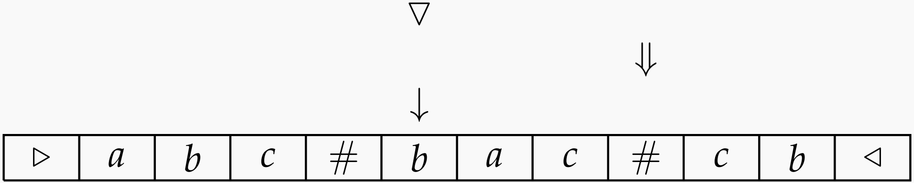


Output: $\quad abcabc\#bac$

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$



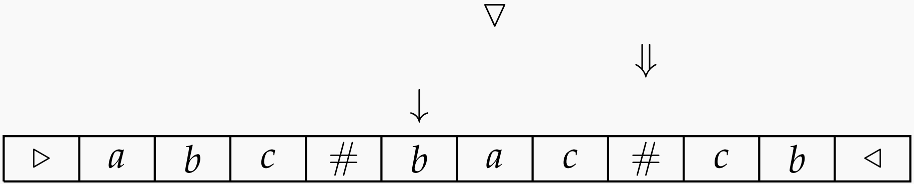
Output: $\quad abcabc\#bac$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \ldots \# w_n \longmapsto (w_0)^n \# \ldots \# (w_n)^n$



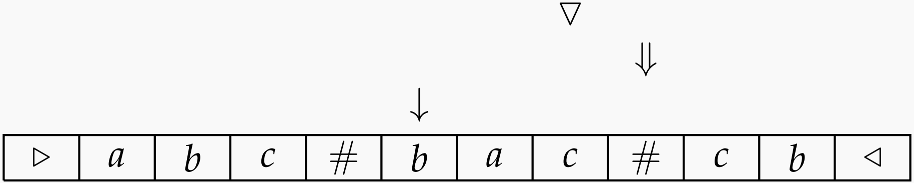
Output: $\quad abcabc\#bacb$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$



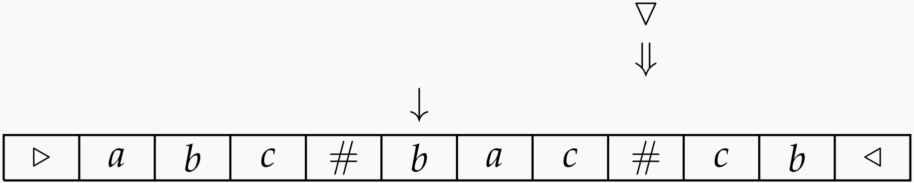
Output: $\quad abcabc\#bacba$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \ldots \# w_n \longmapsto (w_0)^n \# \ldots \# (w_n)^n$



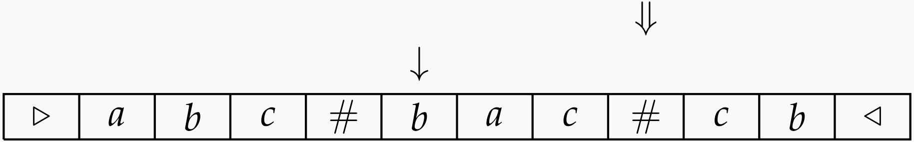
Output: $\quad abcabc\#bacbac$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads (“pebbles”)

“Inner squaring” $\quad \texttt{innsq}\colon w_0\# \ldots \# w_n \longmapsto (w_0)^n \# \ldots \# (w_n)^n$



Output: $\quad abcabc\#bacbac$

Pebble transducers

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $abcabc\#bacbac$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\#\ldots\#w_n \longmapsto (w_0)^n\#\ldots\#(w_n)^n$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $\quad abcabc\#bacbac$

Pebble transducers

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\;\;\texttt{innsq}\colon w_0\#\ldots\#w_n \longmapsto (w_0)^n\#\ldots\#(w_n)^n$

					$\downarrow$						$\Downarrow$
$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $\;\; abcabc\#bacbac$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $\quad abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\#\ldots\#w_n \longmapsto (w_0)^n\#\ldots\#(w_n)^n$

						$\downarrow$					
$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $\quad abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\#\ldots\#w_n \longmapsto (w_0)^n\#\ldots\#(w_n)^n$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $\quad abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\#\ldots\#w_n \longmapsto (w_0)^n\#\ldots\#(w_n)^n$

								↓			
$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $\quad abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\#\ldots\#w_n \longmapsto (w_0)^n\#\ldots\#(w_n)^n$

									$\downarrow$		
$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $\quad abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\#\ldots\#w_n \longmapsto (w_0)^n\#\ldots\#(w_n)^n$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $\quad abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

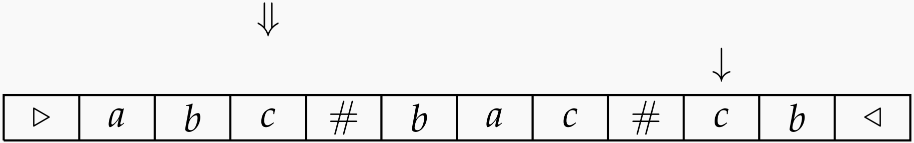
Output: $\quad abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$



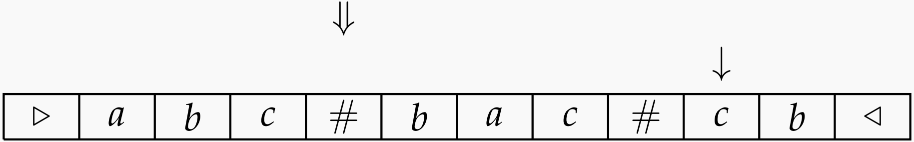
Output: $\quad abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\#\ldots\#w_n \longmapsto (w_0)^n\#\ldots\#(w_n)^n$



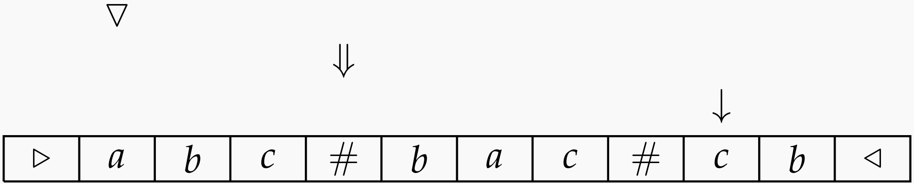
Output: $\quad abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$



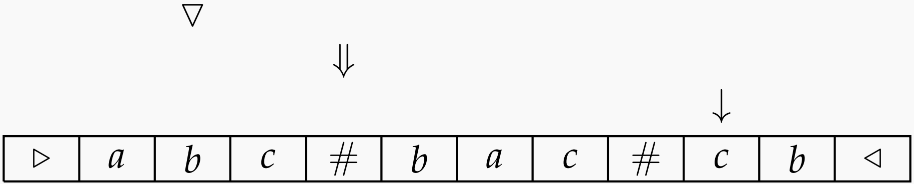
Output: $\quad abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads (“pebbles”)

“Inner squaring” $\quad \texttt{innsq}\colon w_0\# \ldots \# w_n \longmapsto (w_0)^n \# \ldots \# (w_n)^n$



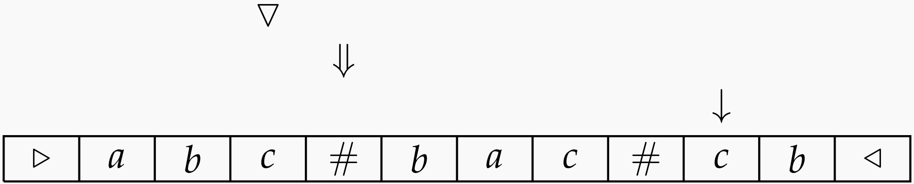
Output: $\quad abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\mathtt{innsq}\colon w_0\#\ldots\#w_n \longmapsto (w_0)^n\#\ldots\#(w_n)^n$



Output: $abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads (“pebbles”)

“Inner squaring” $\texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

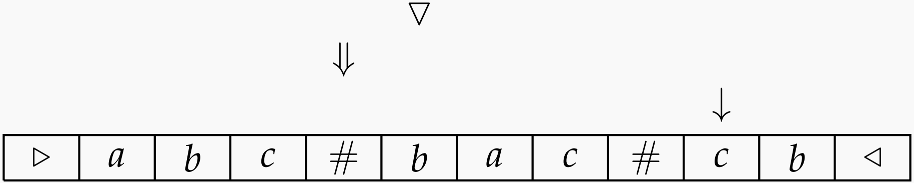
Output: $abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\texttt{innsq}: w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$



Output: $abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$

$\triangledown$

$\Downarrow$

$\downarrow$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $\quad abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$

				$\Downarrow$			$\triangledown$		$\downarrow$		
$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

Output: $\quad abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\#\ldots\#w_n \longmapsto (w_0)^n\#\ldots\#(w_n)^n$

								$\triangledown$			
				$\Downarrow$							
									$\downarrow$		
$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

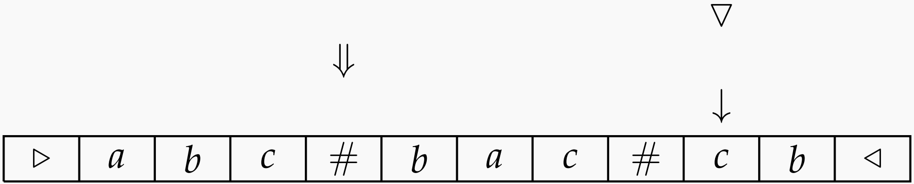
Output: $\quad abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\texttt{innsq}\colon w_0\# \ldots \# w_n \longmapsto (w_0)^n\# \ldots \#(w_n)^n$



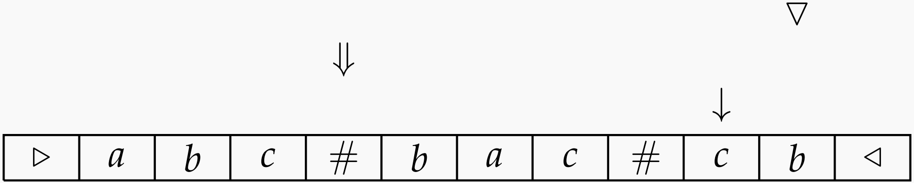
Output: $abcabc\#bacbac\#$

Pebble transducers

Polyregular functions = computed by *k*-pebble transducers ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$



Output: $abcabc\#bacbac\#c$

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \dots \#w_n \longmapsto (w_0)^n\# \dots \#(w_n)^n$

$\triangleright$	a	b	c	$\#$	b	a	c	$\#$	c	b	$\triangleleft$

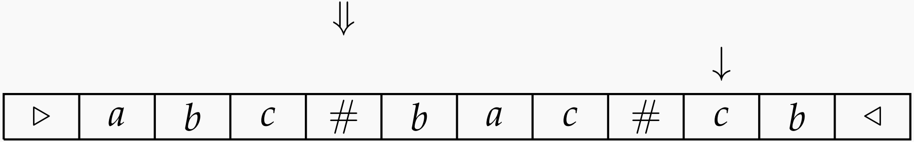
Output: $\quad abcabc\#bacbac\#cb$

Pebble transducers

Polyregular functions = computed by k-pebble transducers ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$



Output: *abcabc#bacbac#cb*

Pebble transducers

Polyregular functions = computed by <u>k-pebble transducers</u> ($k \geq 1$)

Finite states + *stack* of height $\leqslant k$ of two-way heads ("pebbles")

"Inner squaring" $\quad \texttt{innsq}\colon w_0\# \dots \# w_n \longmapsto (w_0)^n \# \dots \# (w_n)^n$

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Output: $\quad abcabc\#bacbac\#cb$

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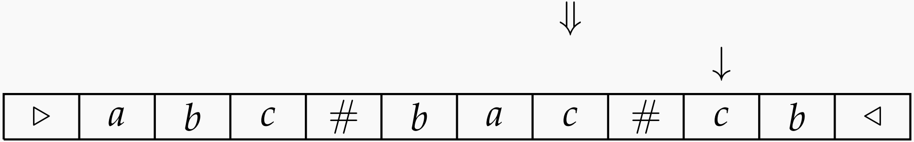

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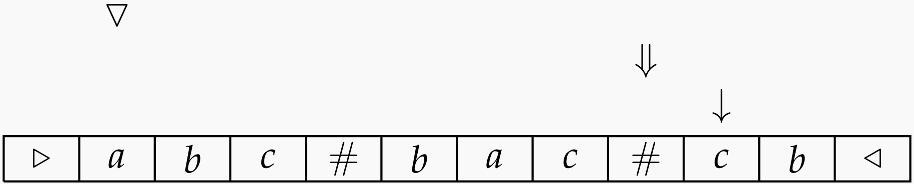

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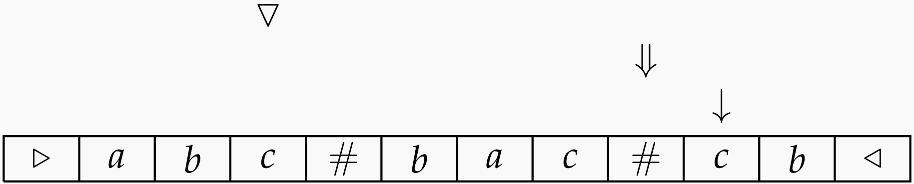

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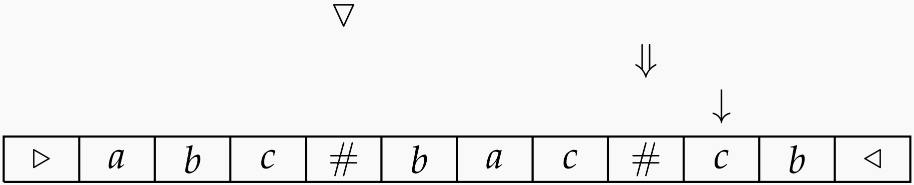

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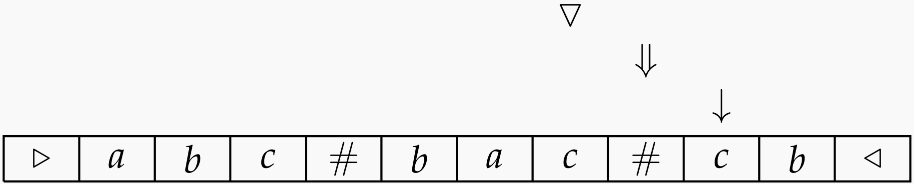

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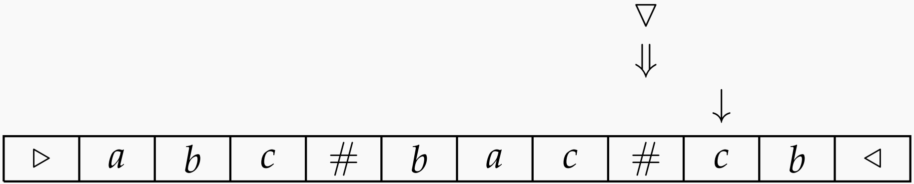

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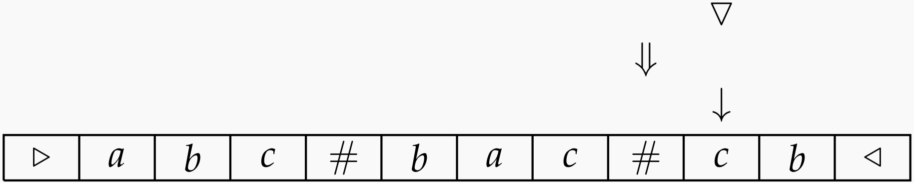

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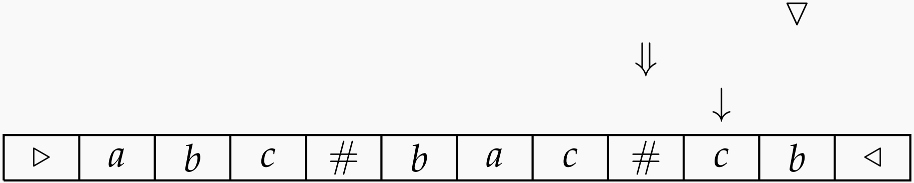

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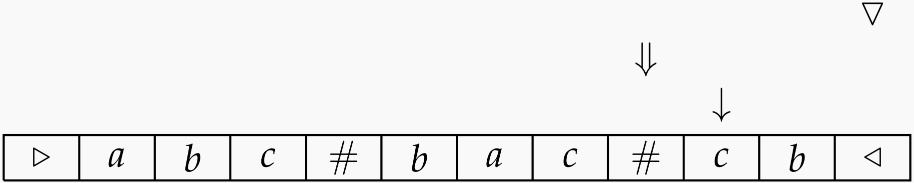

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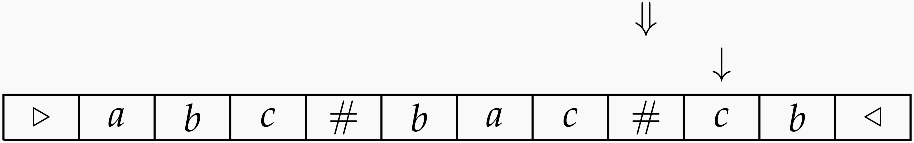

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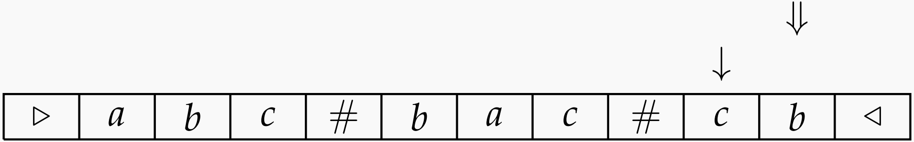

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From "regular" to "polyregular"

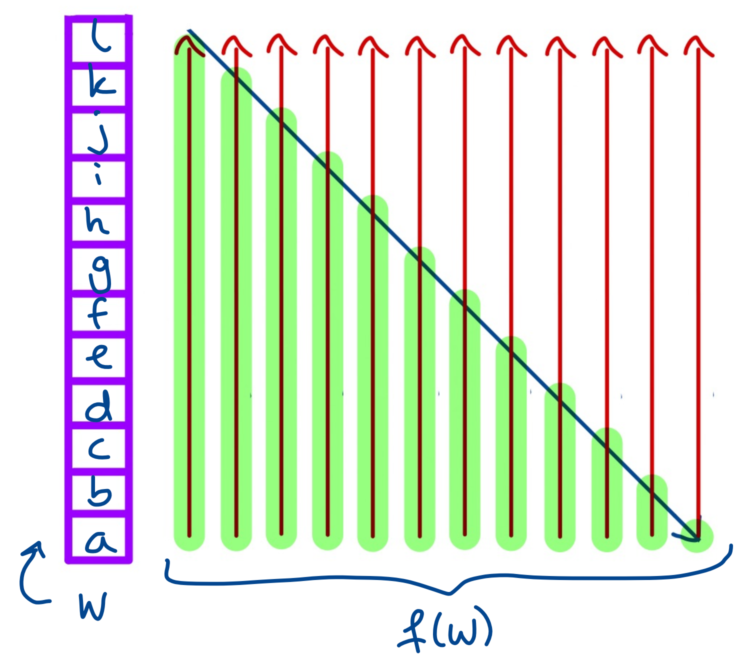
Regular functions = functions computed by deterministic 2-way transducers

For regular functions, the output length is always at most linear in the input length: $f(|w|) = O(|w|)$ ⇒ linear growth rate

How can we modify the model to go <u>beyond</u> linear growth?

We equip the 2-way transducers with multiple reading heads, which can also serve as markers ("pebbles").

→ Tito's pebble transducer simulation ←



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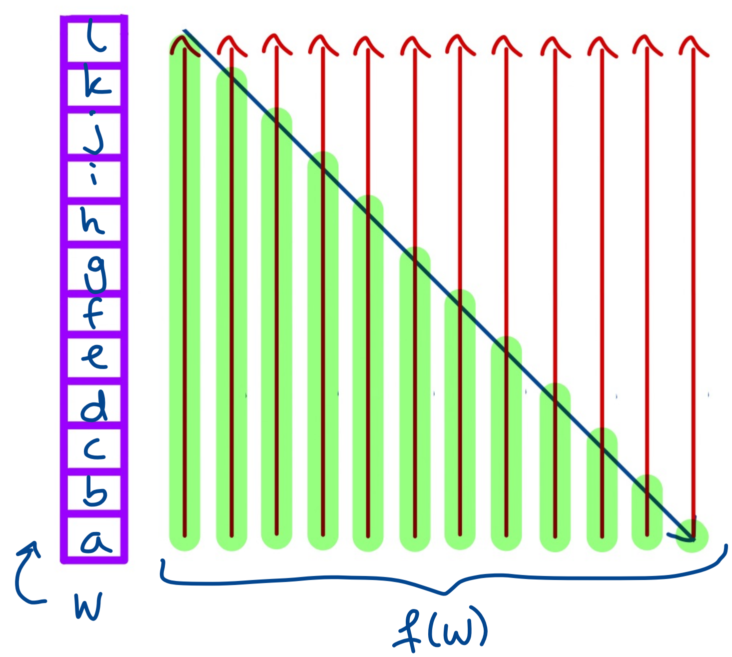
Configurations now depend on all k reading heads. Thus, $f(|w|) = \mathcal{O}(|w|^k)$



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Idea: introduce multiple reading heads to enable polynomial growth

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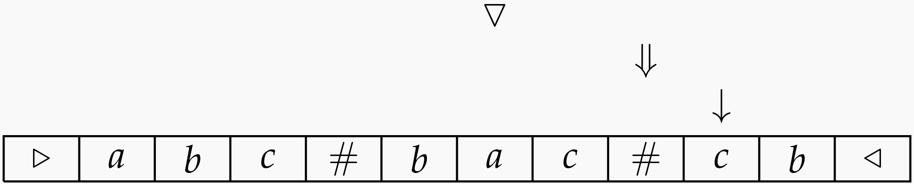

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  - s-to-s MSO interpretations

This talk!

Polyregular functions

- map strings to strings

abcd ↦ a b c d a b c a b a

Polyregular functions

- map strings to strings $\quad abcd \mapsto a\,b\,c\,d\;\; a\,b\,c\;\; a\,b\;\; a$
- positions in the output string "are" k-tuples of positions in the input string (+ finite state)

Polyregular functions

- map strings to strings $\qquad abcd \mapsto \overset{\overset{1}{4}}{a}\ b\ c\ d\ \ a\ b\ c\ \ a\ b\ \ a$

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$$abcd \mapsto \begin{array}{c|c|c|c|c|c|c|c|c} \substack{1\\4} & \substack{2\\4} & \substack{3\\4} & \substack{4\\4} & \substack{1\\3} & \substack{2\\3} & \substack{3\\3} & \substack{1\\2} & \substack{2\\2} \\ a & b & c & d & a & b & c & a & b \end{array} \; a$$

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$abcd \mapsto$

1	2	3	4	1	2	3	1	2	1
4	4	4	4	3	3	3	2	2	1
a	b	c	d	a	b	c	a	b	a

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$$abcd \mapsto \begin{array}{c|c|c|c|c|c|c|c|c|c} \overset{1}{4} & \overset{2}{4} & \overset{3}{4} & \overset{4}{4} & \overset{1}{3} & \overset{2}{3} & \overset{3}{3} & \overset{1}{2} & \overset{2}{2} & \overset{1}{1} \\ a & b & c & d & a & b & c & a & b & a \end{array}$$

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for-programs are of the shape

```
for i = n to 1
    for j = 1 to n
        if j ≤ i output w(j)
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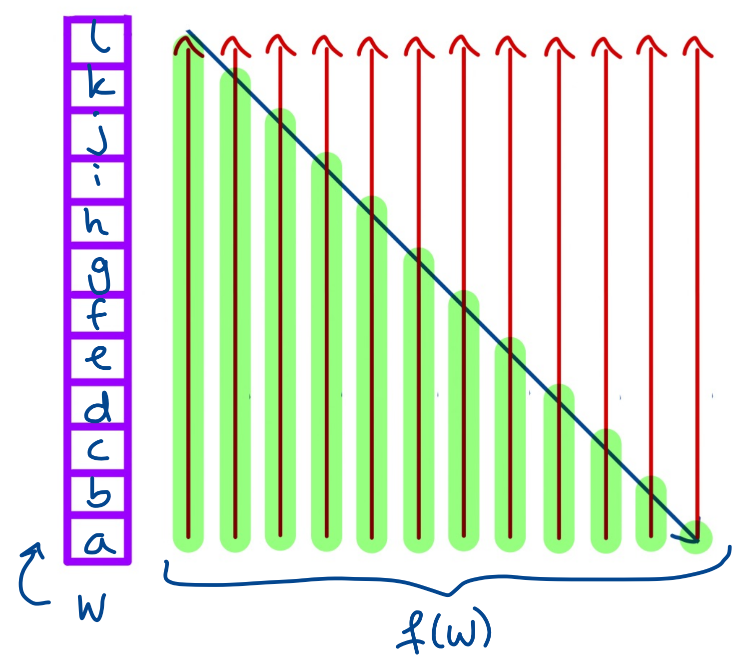
for-loops correspond to spawned pebbles (≈ position markers) in the transducers.

The pebbles obey a stack discipline.



Polyregular functions - Logical characterisation

Concatenation of prefixes

$$abcd \mapsto \overset{\overset{1}{4}}{a} \mid \overset{\overset{2}{4}}{b} \mid \overset{\overset{3}{4}}{c} \mid \overset{\overset{4}{4}}{d} \mid \overset{\overset{1}{3}}{a} \mid \overset{\overset{2}{3}}{b} \mid \overset{\overset{3}{3}}{c} \mid \overset{\overset{1}{2}}{a} \mid \overset{\overset{2}{2}}{b} \mid \overset{\overset{1}{1}}{a}$$

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We can describe the output via

- a domain formula $\varphi_{dom}(i,j) = j \leq i$
- a total-order formula $\varphi_{\leq}(i,j,i',j') = (i \geq i') \vee ((i = i') \wedge (j \leq j'))$
- label formulas $\varphi_a(i,j) = a(j)$

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Together, the formulae describe a 2-dim. string-to-string interpretation.

MSO transductions for regular functions = 1-dimensional case

Polyregular functions - Logical characterisation

Concatenation of prefixes

$$abcd \mapsto$$

1	2	3	4	1	2	3	1	2	1
4	4	4	4	3	3	3	2	2	1
a	b	c	d	a	b	c	a	b	a

```
for i:=n to 1
   for j:=1 to n
      if j≤i output w(j)
```

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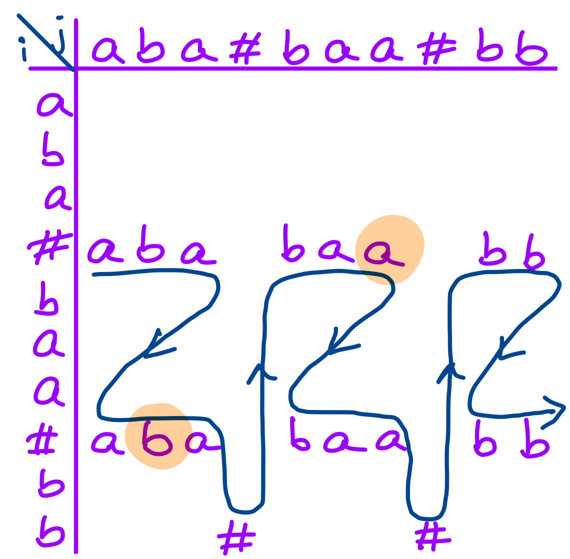

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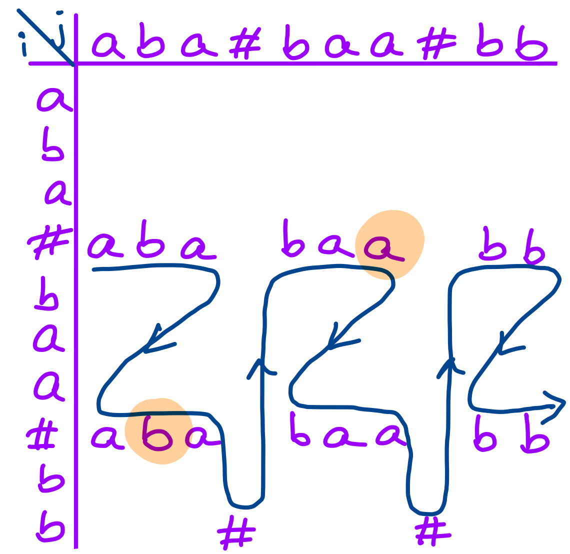

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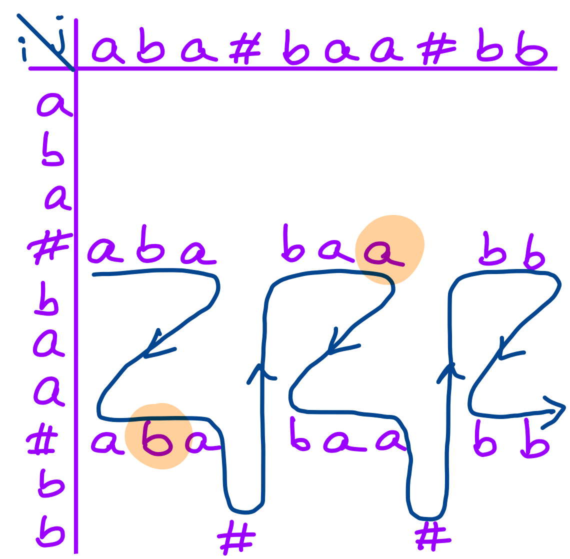

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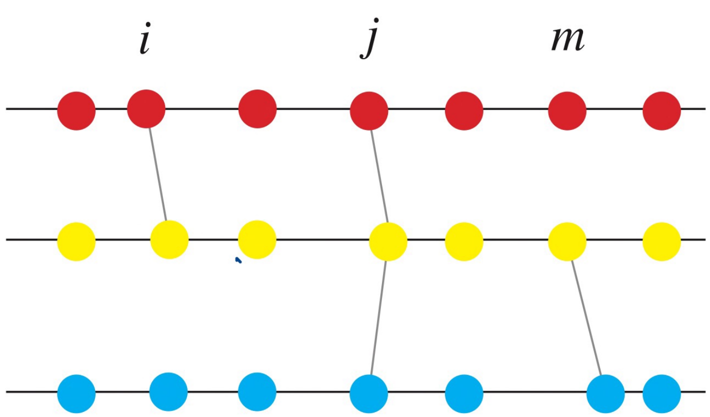

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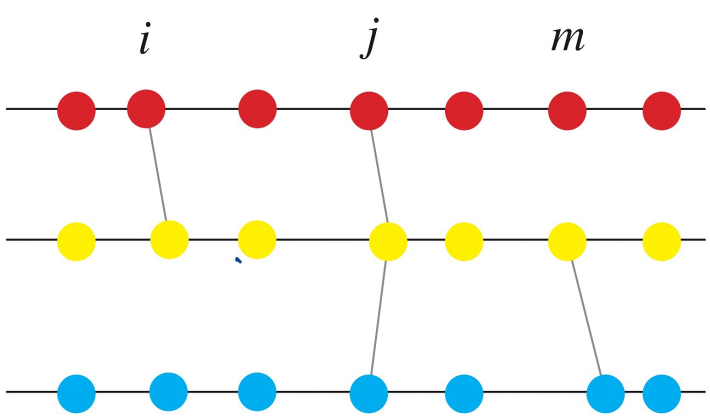

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$\rightsquigarrow$ One coordinate dominates globally!

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The function can be defined via a k-dimensional MSO interpretation.

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Clearly, it holds that : k pebbles $\Rightarrow$ $\mathcal{O}(n^k)$ growth

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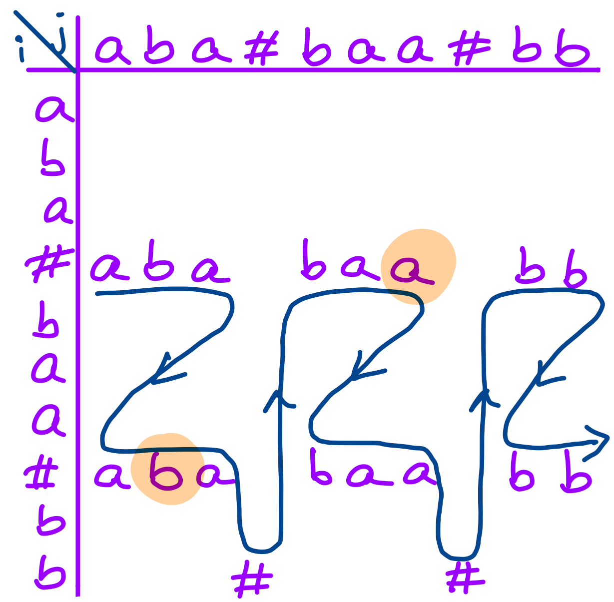

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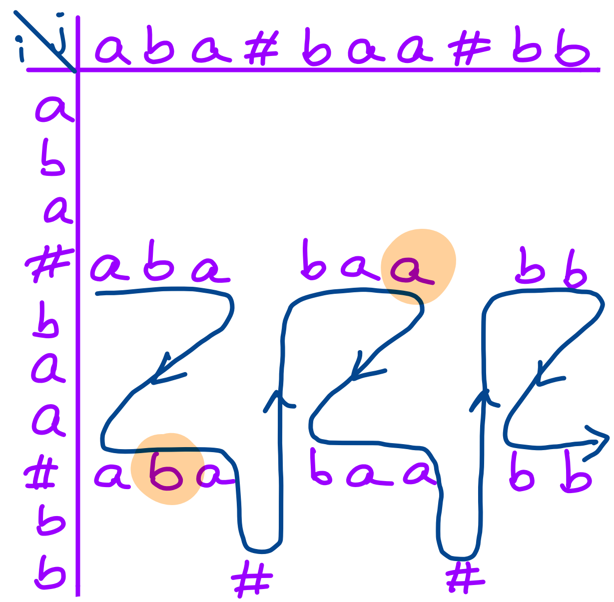

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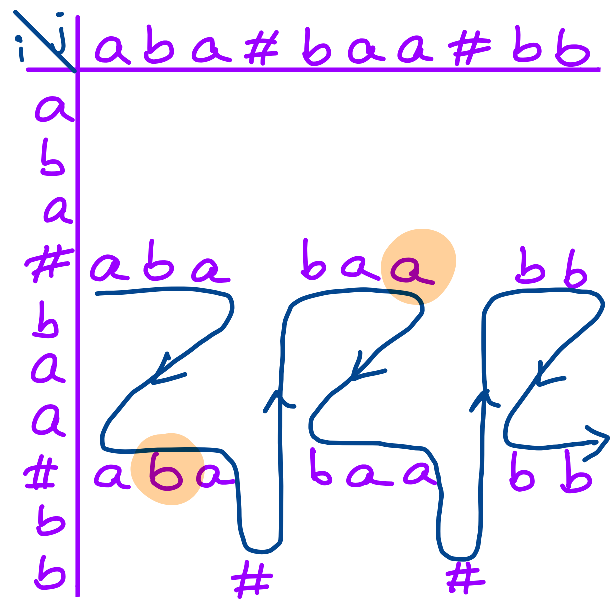

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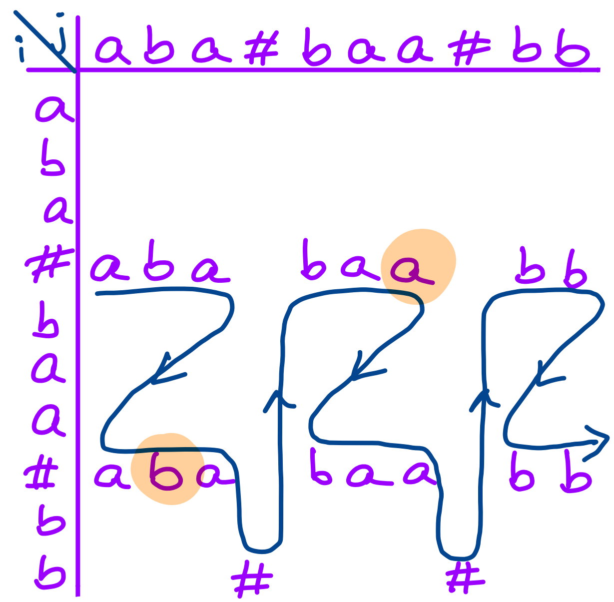

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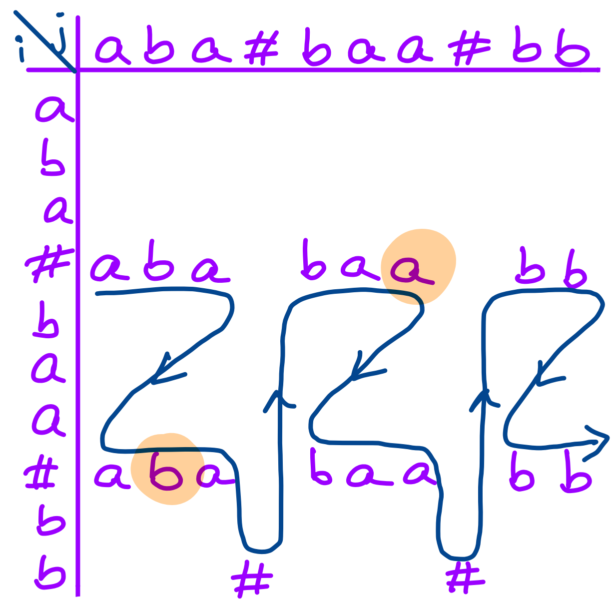

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Our contribution: Easier proofs for the above. [K., Nguyen, Pradic '23]

This talk: Easier proof for innsq $\notin$ Pebble$_2$.

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We apply a pumping argument to L to conclude that it cannot exist.

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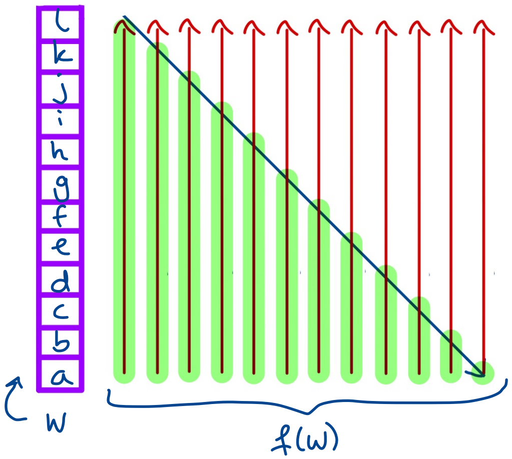

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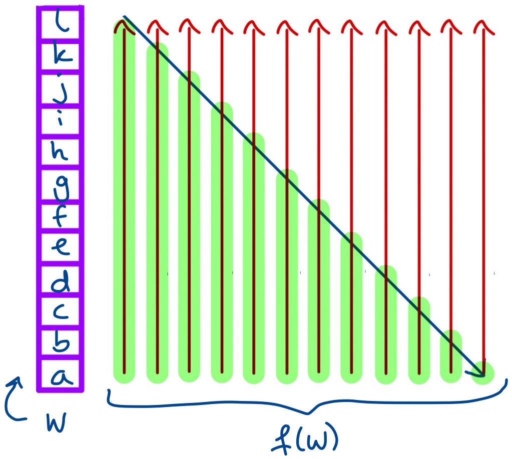

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These having linear growth, the "inner functions" can only produce linearly long infixes.

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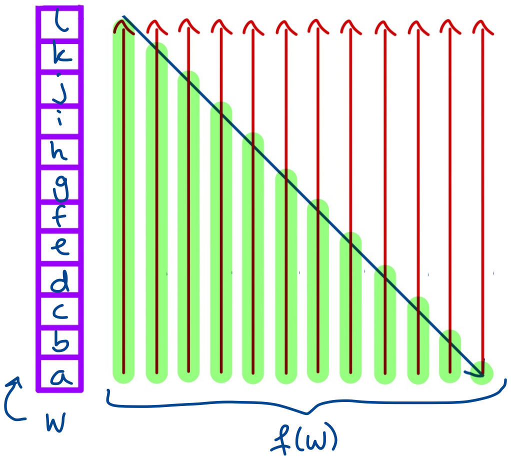

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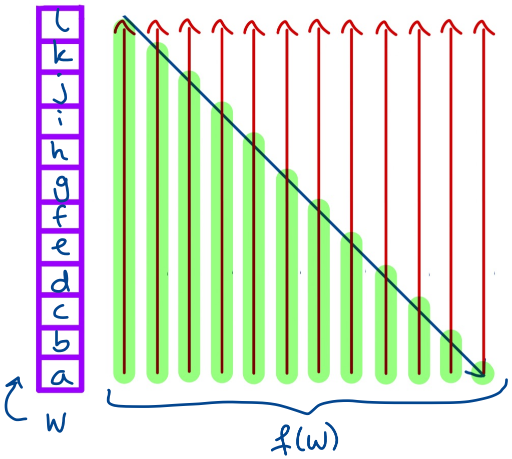

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This way, we construct $L \subseteq b\{a,b\}^*b$ that

- is the image of a regular function
- consists of infixes of elements in innsq$((a^*b\#)^*\#^*)$
- contains for every $N \in \mathbb{N}$ a $b\underbrace{a\ldots a}_{\text{all a-blocks have length } n \geq N}b\ldots ba\ldots ab$ with at least N b's.

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$L \subseteq b\{a,b\}^* b$ is a regular image that

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There are $k, K \in \mathbb{N}$ such that every $w \in L$ with $|w| \geq K$ has a decomposition $w = u_0 v_1 u_1 \ldots v_k u_k$ with

- $v_i \neq \varepsilon$ for some $i \in \{1, \ldots, k\}$
- $|v_i| \leq K$ for all $i \in \{1, \ldots, k\}$
- $\{u_0 v_1^n \ldots u_{k-1} v_k^n u_k \mid n \in \mathbb{N}\} \subseteq L$

[Rozoy '86] **PUMPING LEMMA**

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For $N := \max\{K, 2k+2\}$, we obtain a $b(a^n b)^r \in L$ with $n \geq N \geq K$, $r \geq N \geq 2k+1$.

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Each v_i contains at most one b. $\Rightarrow$ There is a u_j which contains two b's.

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Case ①: Some v_i contains a b. ↯ Case ②: All v_i are in a^*. ↯

$\square$

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f is polyregular	$:\Longleftrightarrow$	f is computed by a for-program	
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Follow-up work and concepts:

- Polyblind functions – comparisons between pebble positions are not allowed [Nguyễn, Noûs, Pradic '21]

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- $\mathbb{Z}$-polyregular functions – functions $\mathrm{sum} \circ f$, where $f : \Sigma^* \to \{\pm 1\}^*$ is polyregular [Colcombet, Douéneau-Tabot, Lopez '23]

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f is polyregular of growth rate $\mathcal{O}(n^k)$

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For example, innsq requires 3 pebbles, but $\text{innsq}(w) \in \mathcal{O}(|w|^2)$.

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There are also quadratic-growth polyregular functions $(f_k)_{k\in\mathbb{N}}$ where $f_k \notin \mathrm{Pebble}_k$ for each $k \in \mathbb{N}$.

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In [K., Nguyen, Pradic '23], we show that a slightly stronger result actually follows quickly from old results due to [Engelfriet, Maneth '01].