What's Decidable about Discrete Linear Dynamical Systems?

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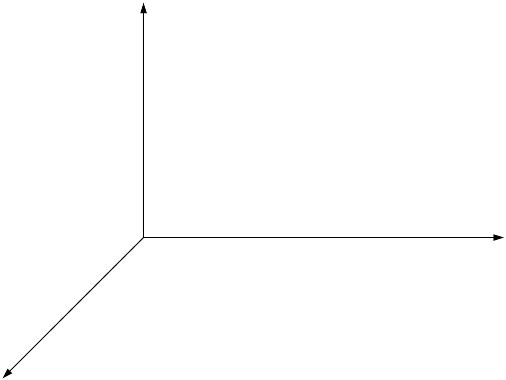
Max Planck Institute for Software Systems

Joint work with: Toghrul Karimov, Shaull Almagor, Ventsi Chonev, Edon Kelmendi, Engel Lefaucheux, Florian Luca, Joris Nieuwveld, David Purser, João Sousa Pinto, Anton Varonka, Markus Whiteland, James Worrell, . . .

Theorietag Automaten und Formale Sprachen
Kaiserslautern, Oktober 2023

Reachability for discrete linear dynamical systems

Ambient space: $\mathbb{R}^d$ ($\mathbb{R}^3$ in this example)
Starting point: $\mathbf{x} \in \mathbb{Q}^d$
Linear transformation: $\mathbf{M} \in \mathbb{Q}^{d \times d}$



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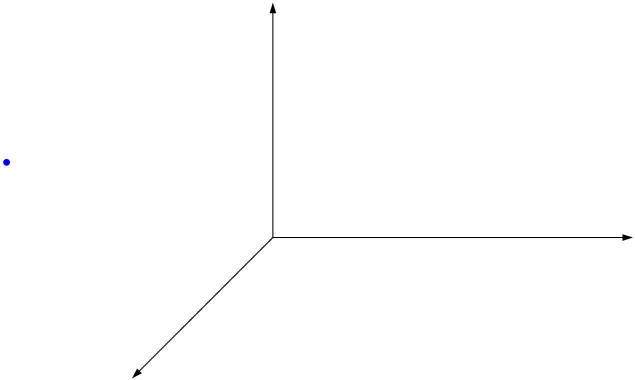

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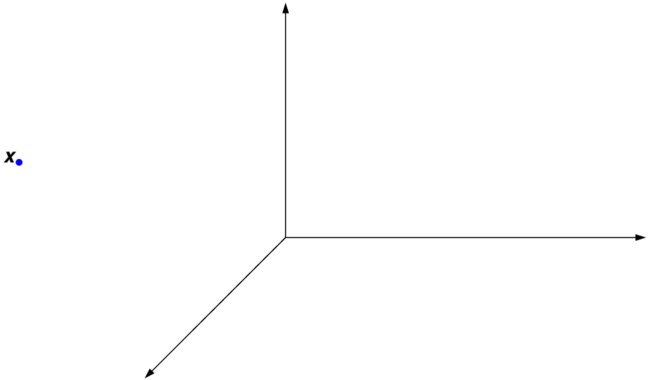

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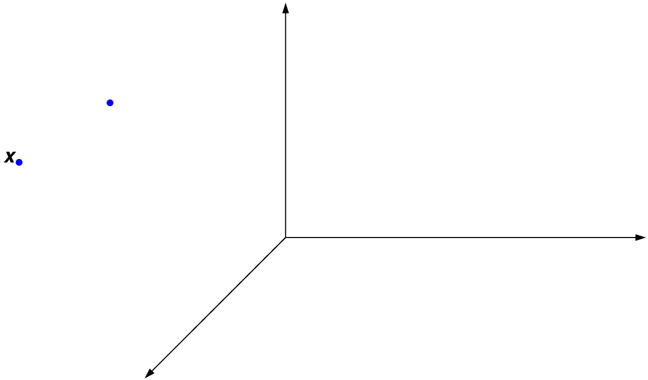

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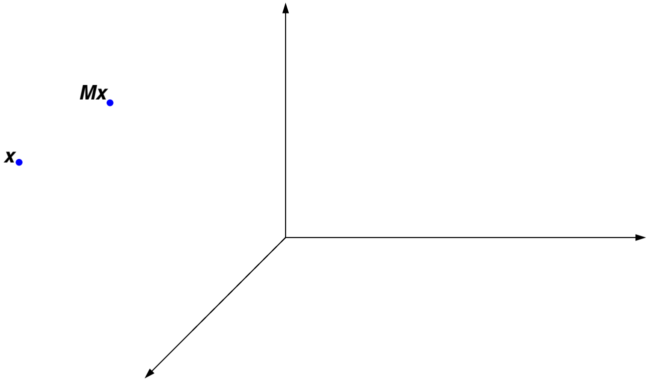

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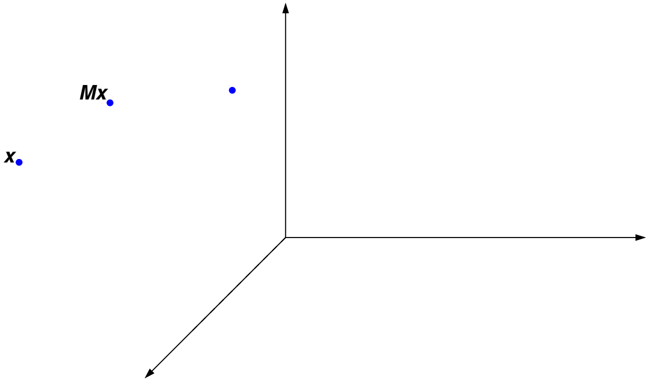

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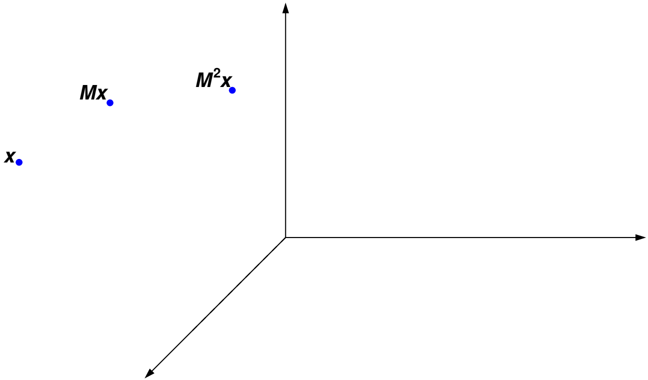

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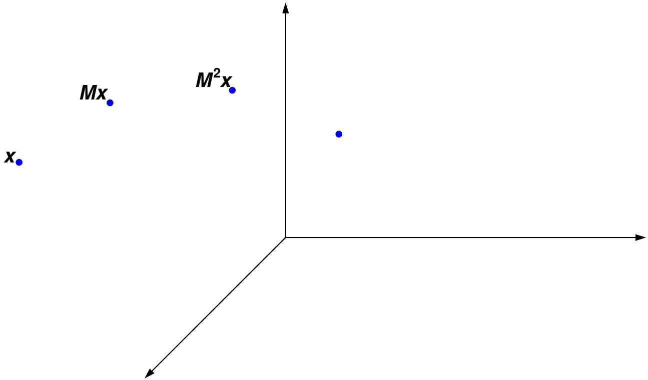

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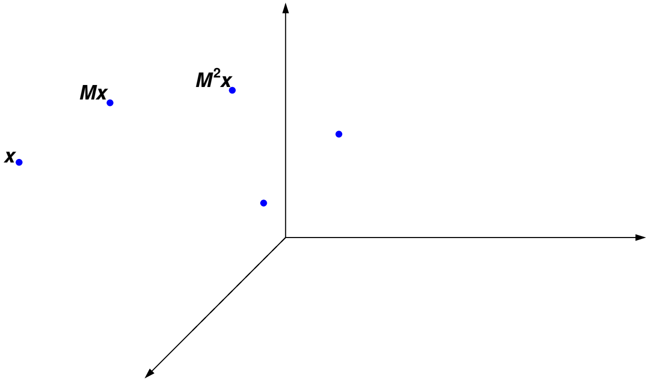

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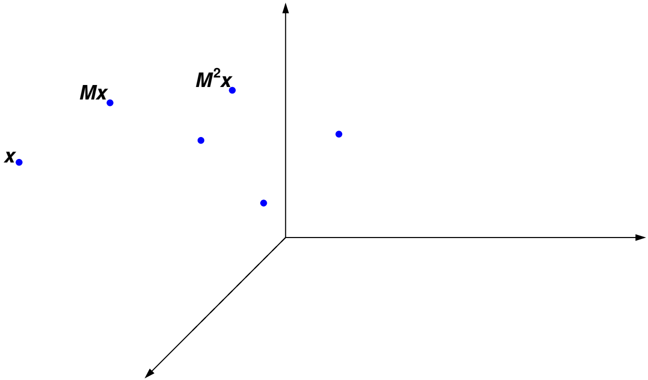

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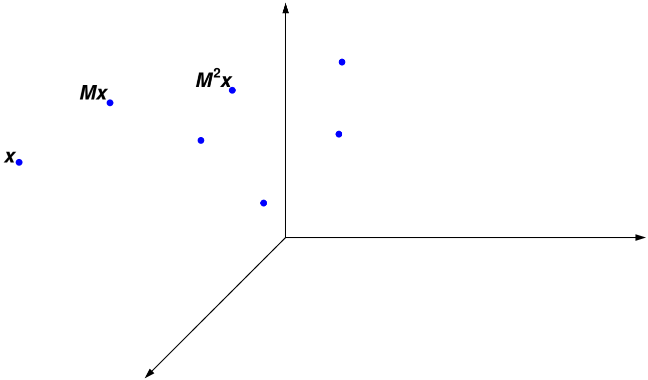

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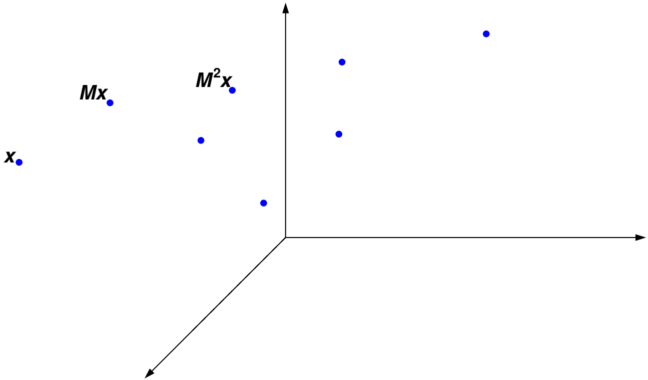

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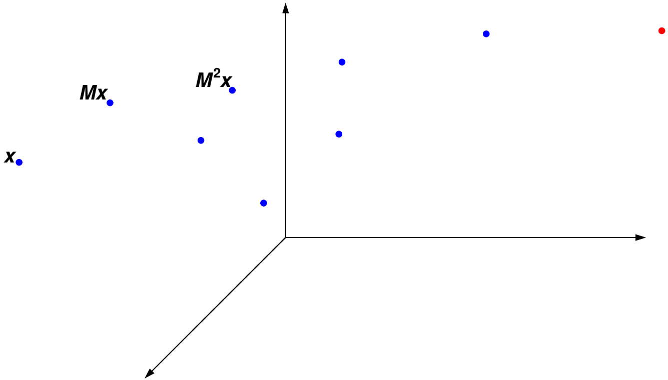

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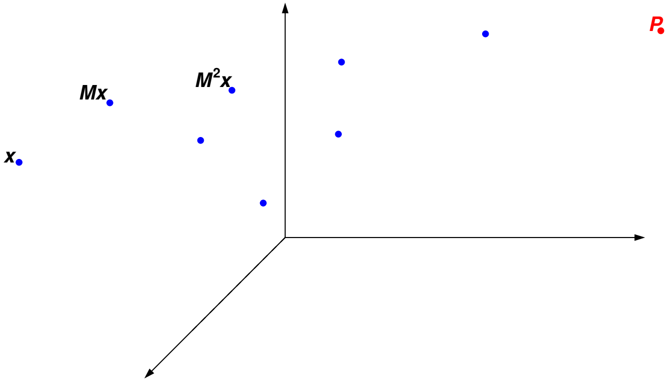

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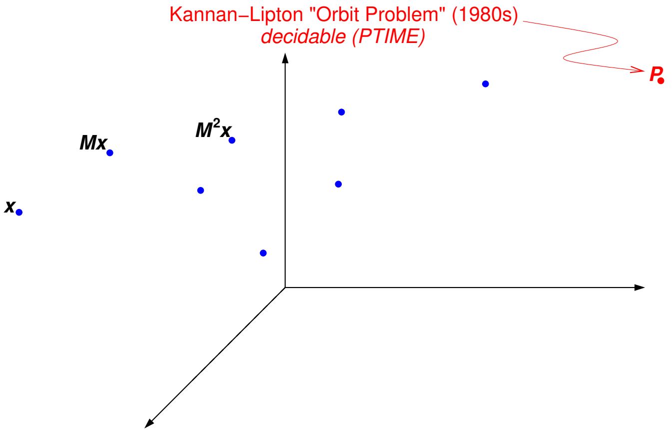

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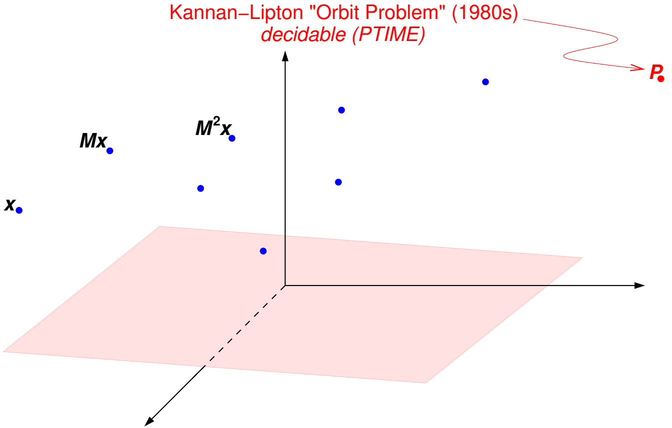

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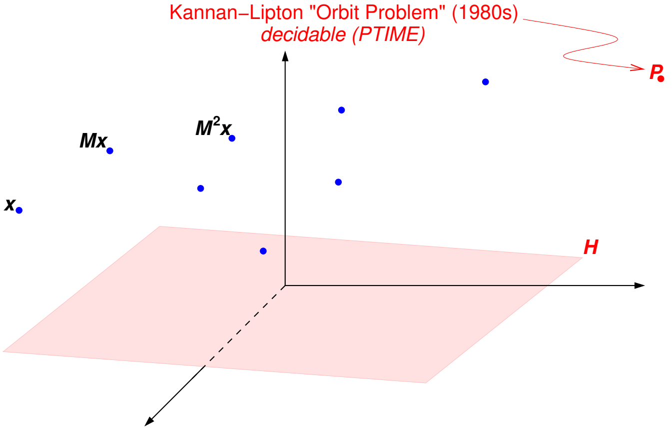

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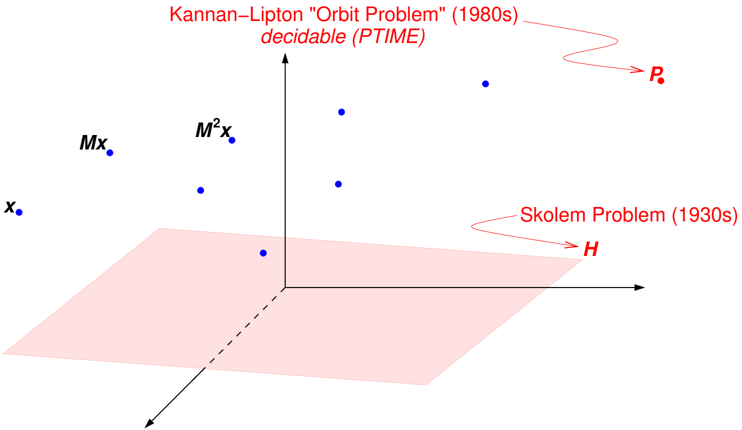

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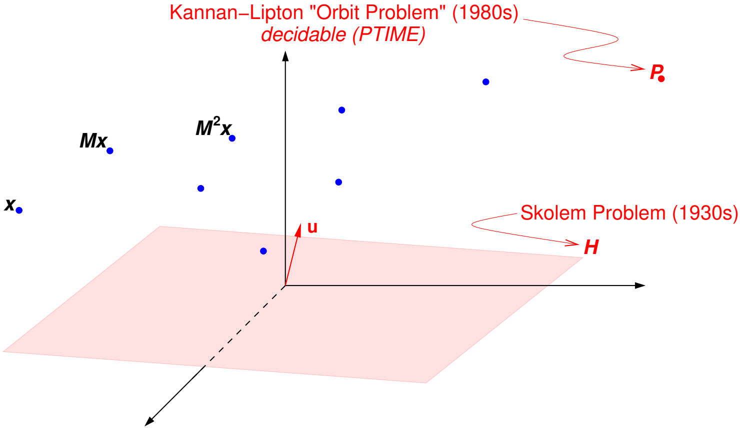

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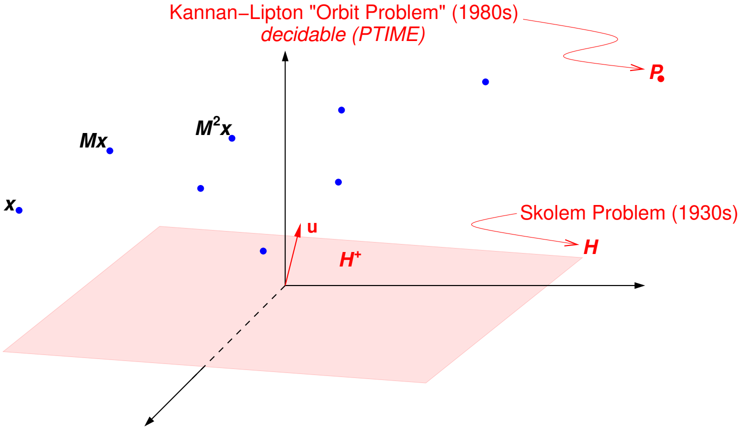

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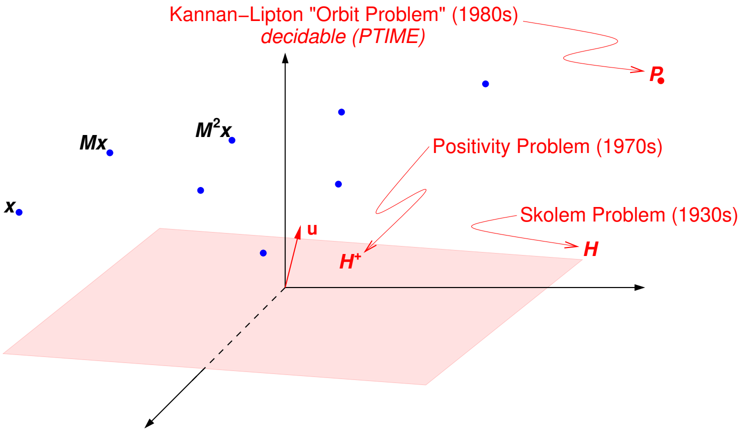
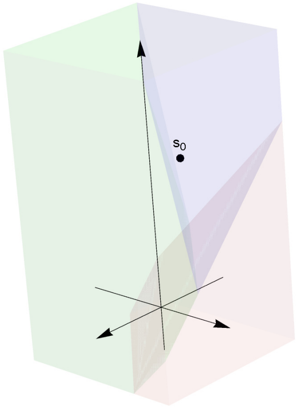


Semialgebraic partitions

Partition $\mathbb{R}^d$ into



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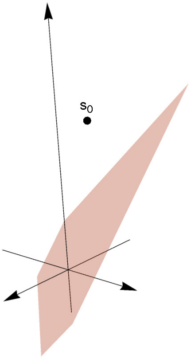
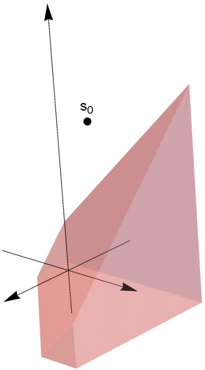
Partition $\mathbb{R}^d$ into S_1



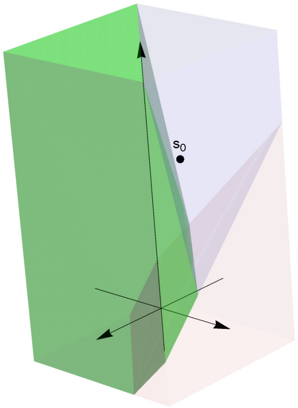
Semialgebraic partitions

Partition $\mathbb{R}^d$ into S_1, S_2



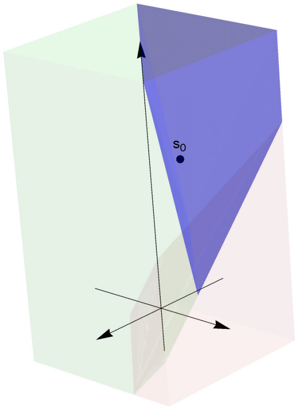
Semialgebraic partitions

Partition $\mathbb{R}^d$ into S_1, S_2, S_3



Semialgebraic partitions

Partition $\mathbb{R}^d$ into S_1, S_2, S_3, S_4



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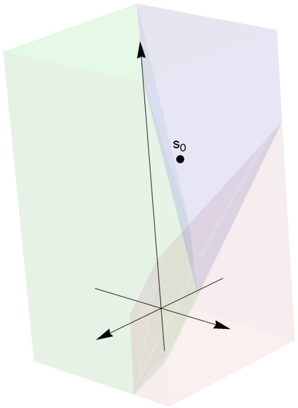

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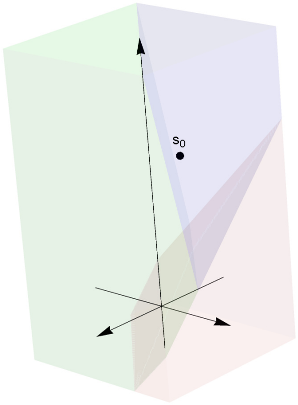

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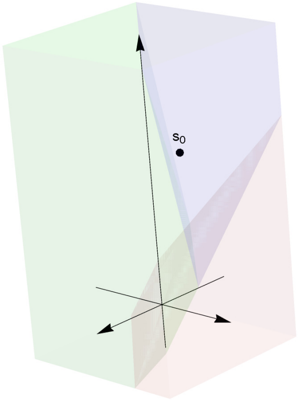

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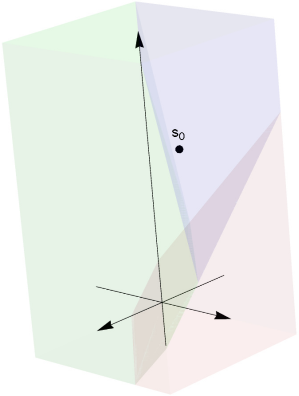

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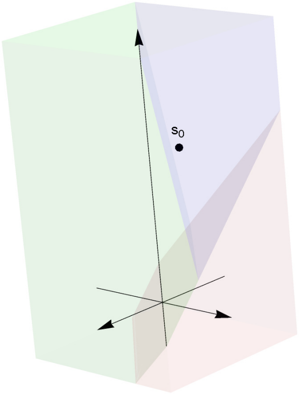

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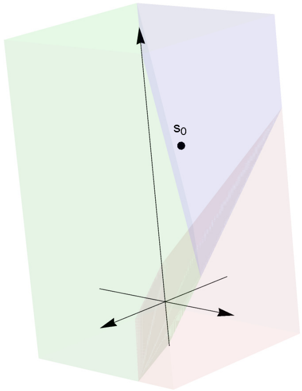

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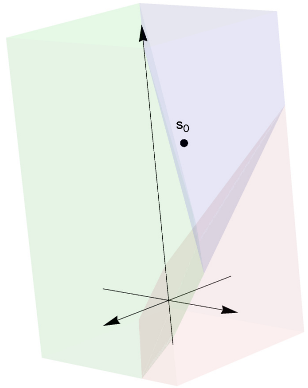

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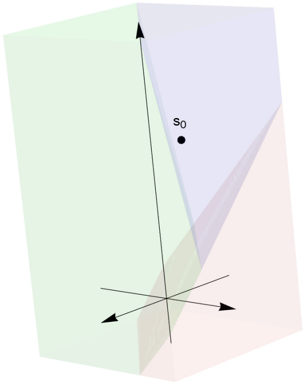

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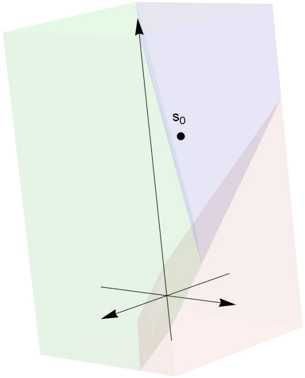

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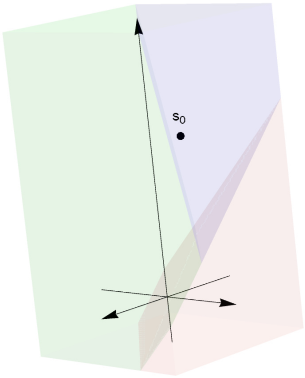

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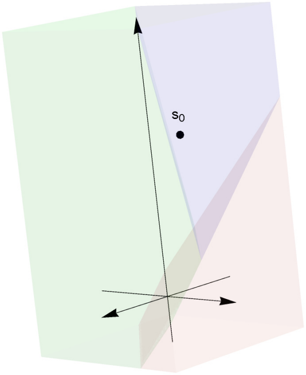

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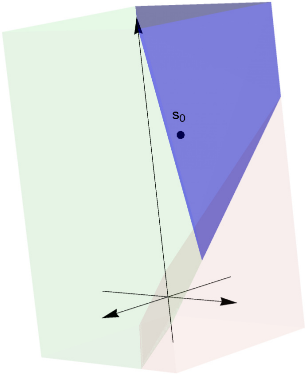

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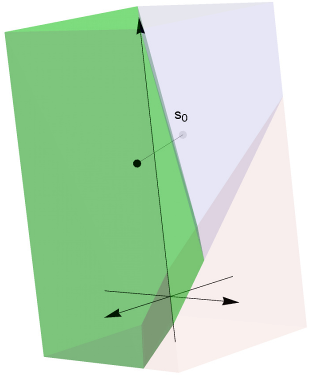

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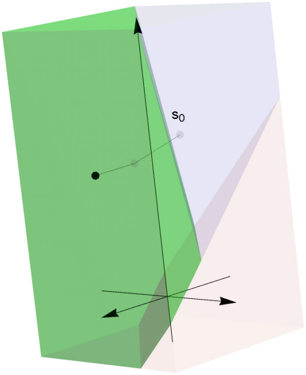

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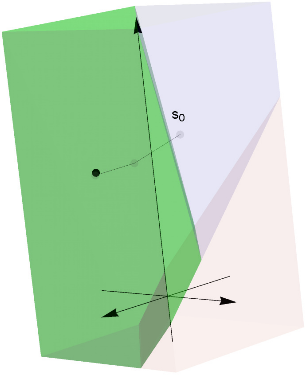

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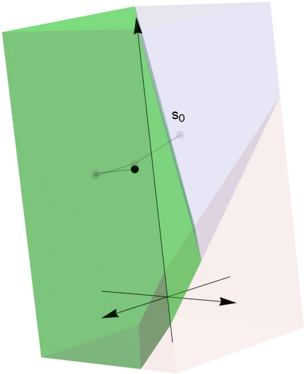

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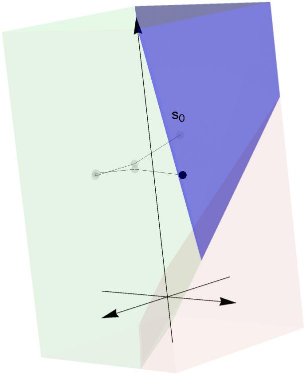

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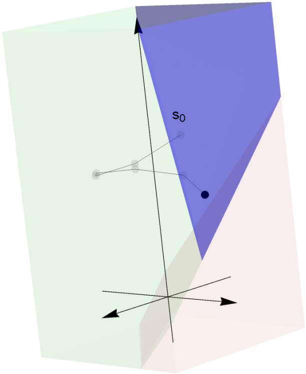

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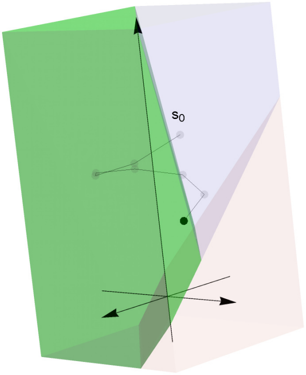

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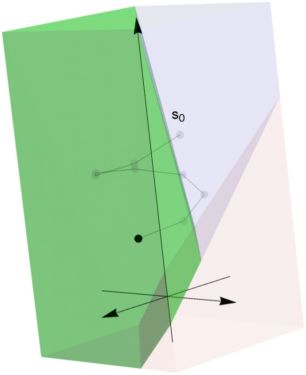

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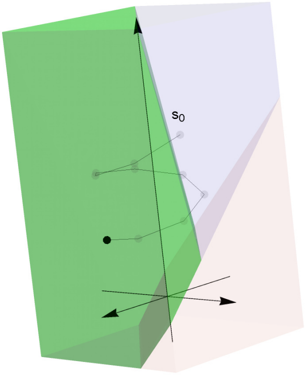

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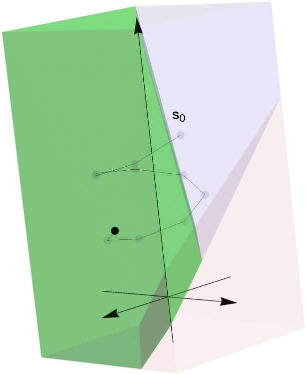

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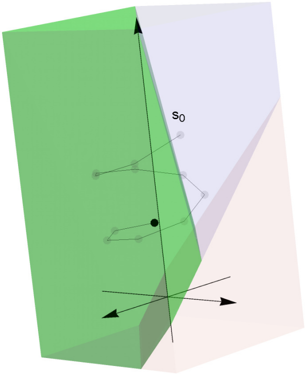

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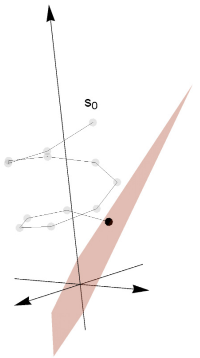

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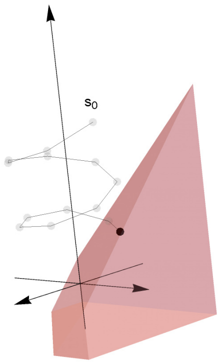

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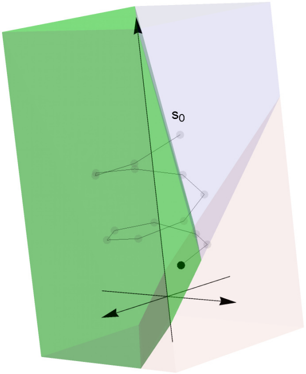

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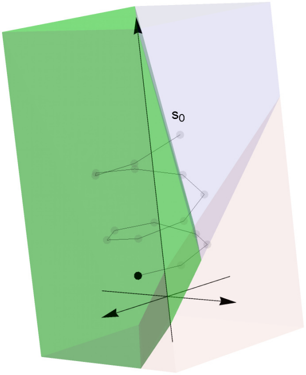

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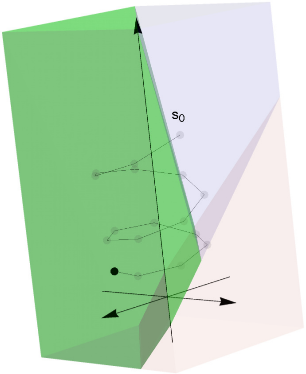

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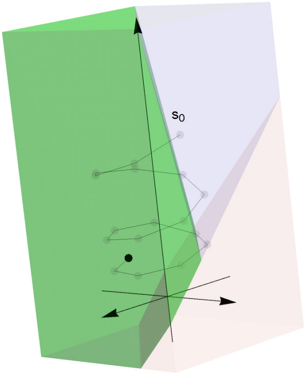

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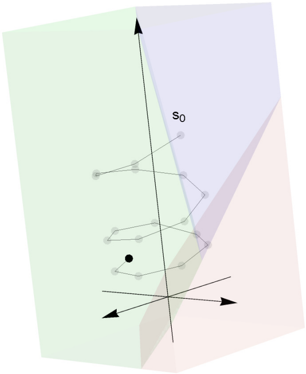

Model checking discrete linear dynamical systems

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generated by (M, s)

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- Deciding ω-Regular Properties on Linear Recurrence Sequences
Almagor, Karimov, Kelmendi, O., Worrell, in *POPL* 2021

- What's Decidable about Linear Loops?
Karimov, Lefaucheux, O., Purser, Varonka, Whiteland, Worrell, in *POPL* 2022

- The Power of Positivity
Karimov, Kelmendi, Nieuwveld, O., Worrell, in *LICS* 2023

- What's Decidable about Discrete Linear Dynamical Systems?
Karimov, Kelmendi, O., Worrell, in *Henzinger Festschrift* 2023

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- The use (or not) of *Skolem and/or Positivity oracles*

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Only (3) and (4)

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Question: Is this specification prefix-independent? **NO!**

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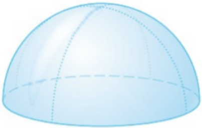

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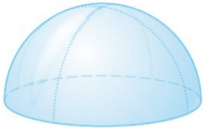

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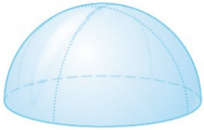

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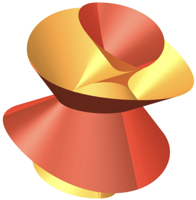

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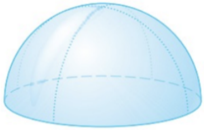

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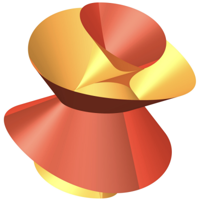

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The class $\mathcal{T}$ comprises all semialgebraic sets that are *either* contained in a three-dimensional subspace of $\mathbb{R}^d$, *or* that have intrinsic dimension at most one. $\mathcal{T}$ is defined to be the smallest such class which is closed under all Boolean operations.

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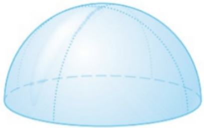

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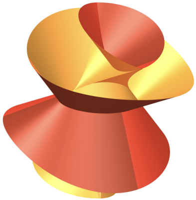

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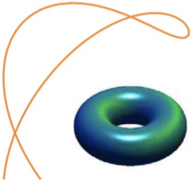
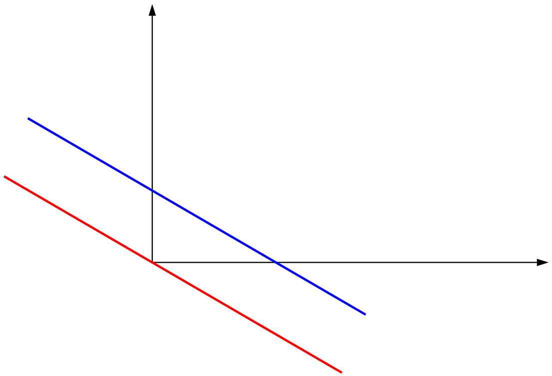


Examples in $\mathbb{R}^2$: two line segments



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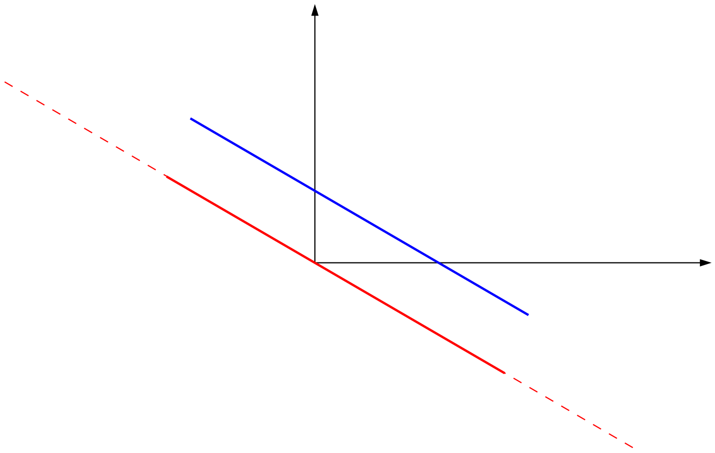

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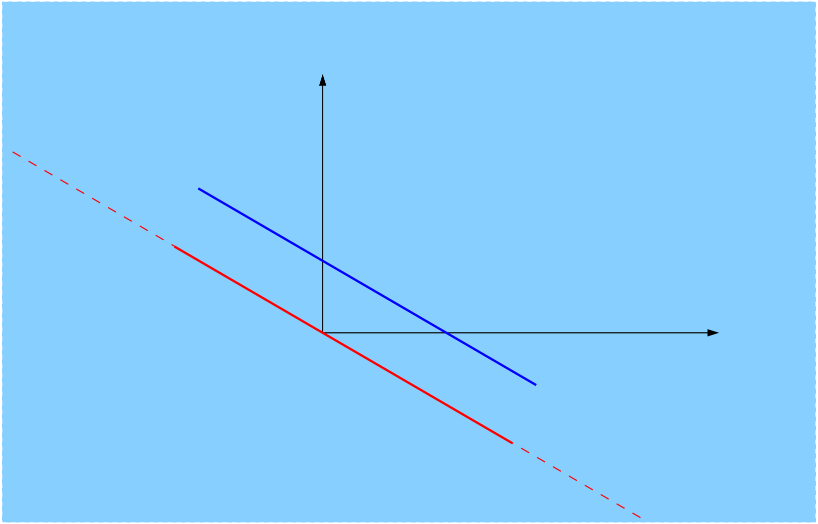

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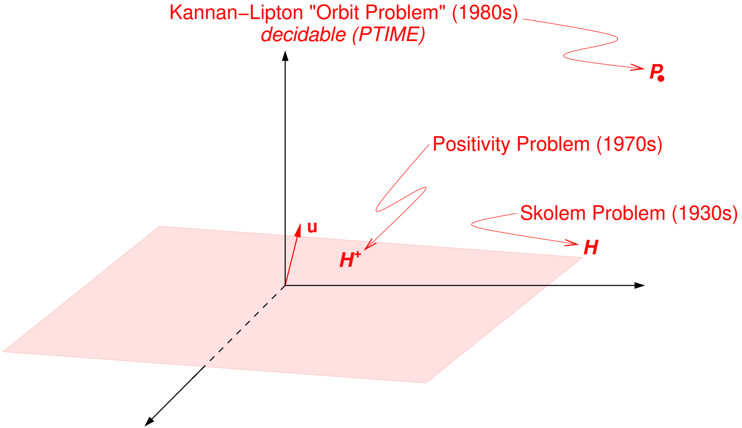

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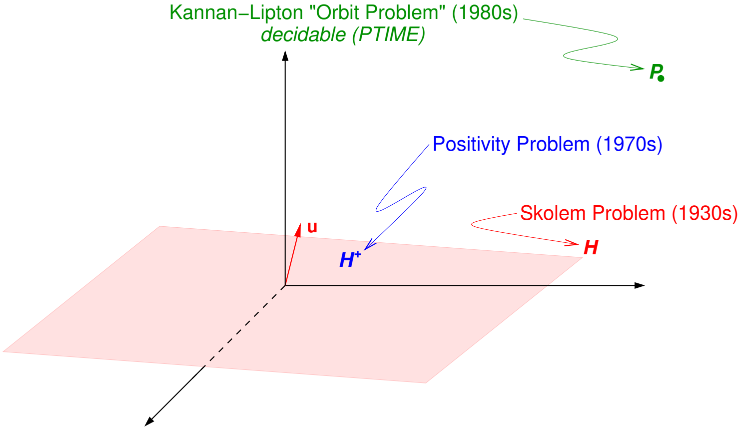

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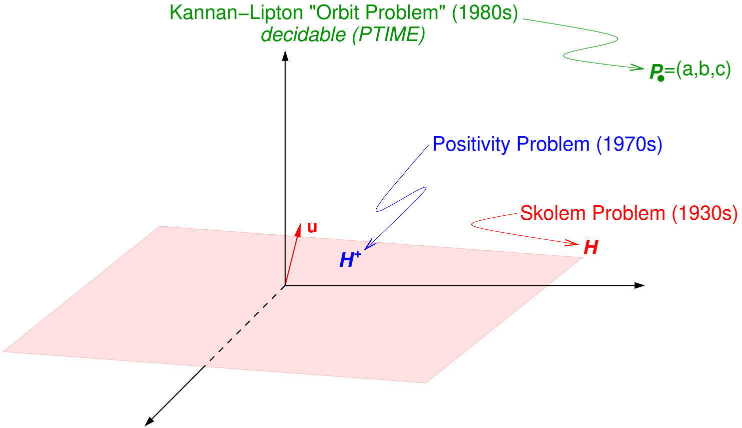

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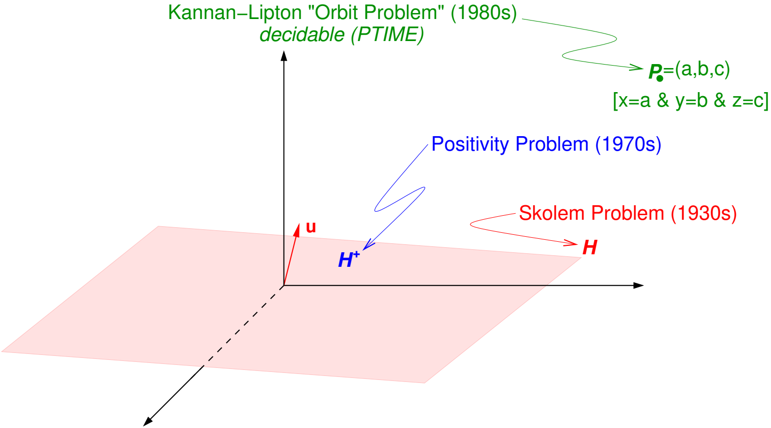

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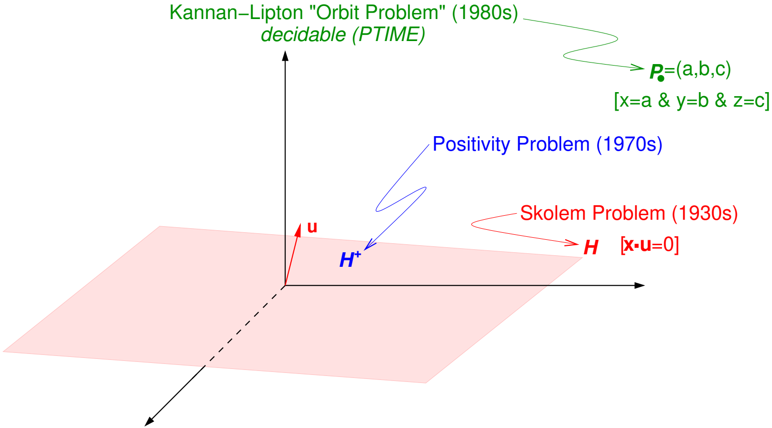

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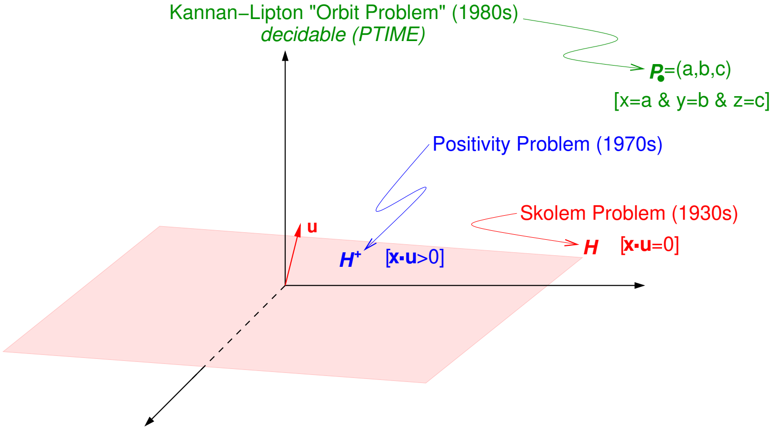
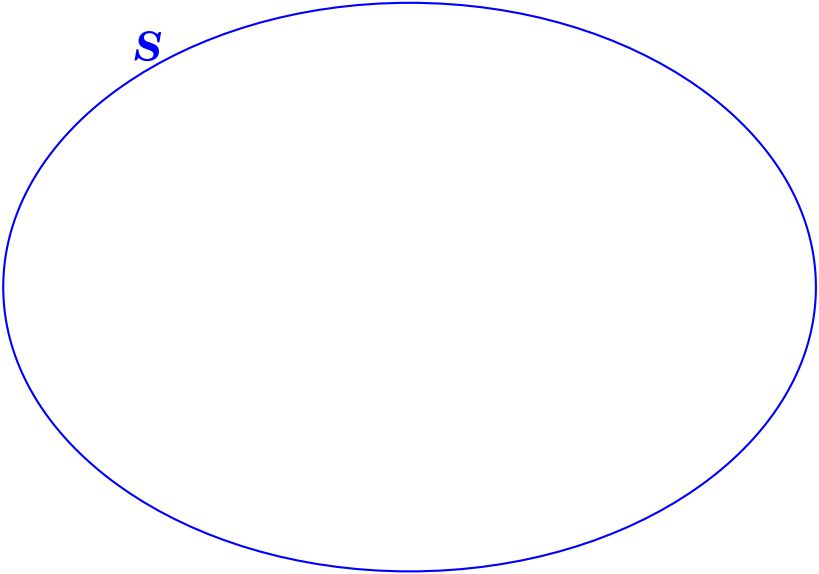


Classes of predicates: summary



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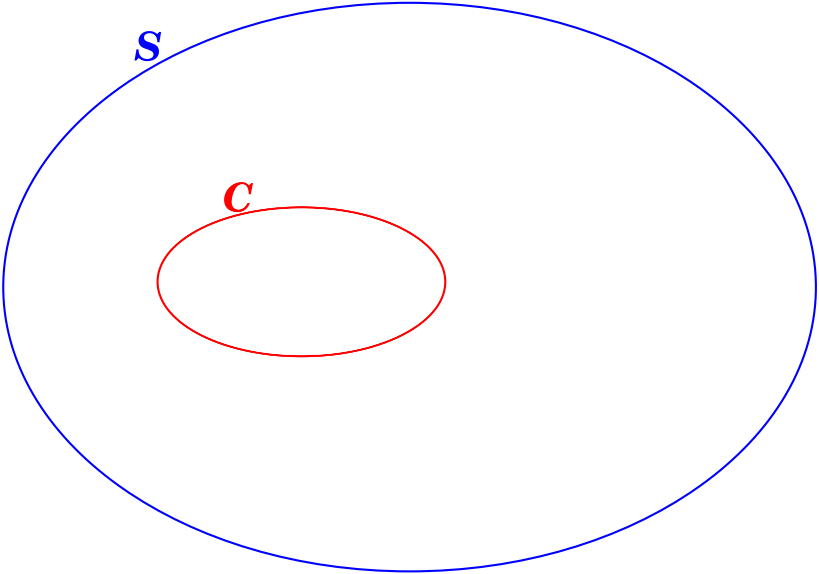

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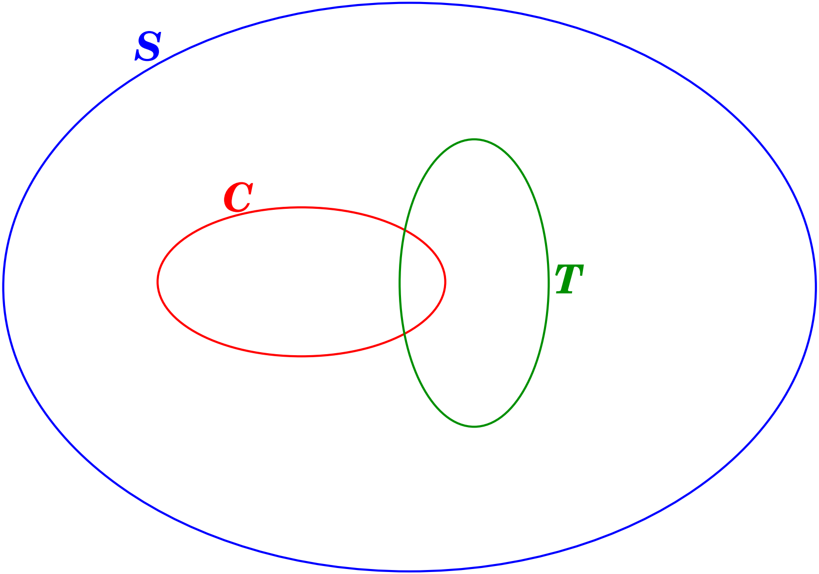

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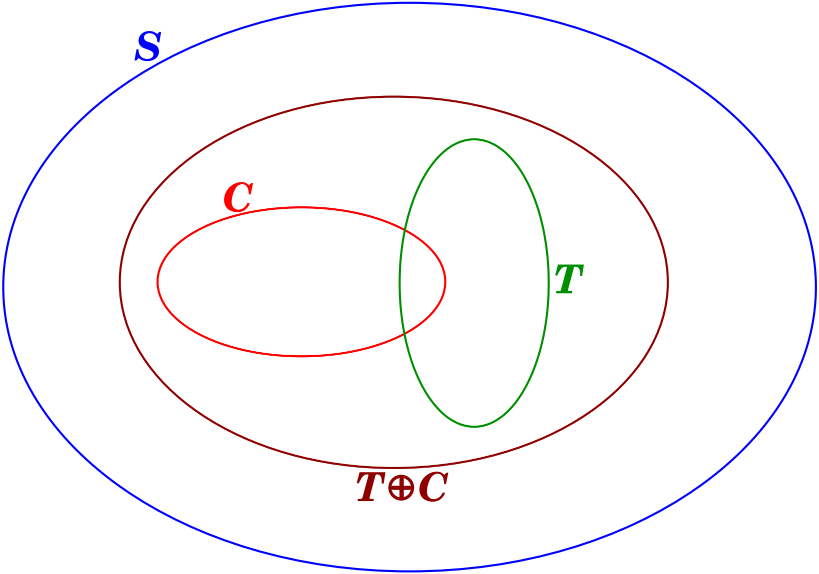

Model checking discrete linear dynamical systems

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	arbitrary LDS	diagonalisable LDS
MSO		
piMSO		

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- unconditional decidability

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Model checking discrete linear dynamical systems

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Model checking discrete linear dynamical systems

- unconditional decidability
- assuming a Skolem oracle

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Model checking discrete linear dynamical systems

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- unconditional decidability
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Moreover, our unconditional decidability algorithm
can produce *correctness certificates!*

Decidability of MSO and extensions

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This is open! But appears very difficult, e.g.

$$\forall x \,.\, \exists y > x \,.\, P(y) \wedge P(y+2)$$

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Theorem (Semënov, 1984)

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A word w is (effectively) almost-periodic if for every finite word u, we can bound the gaps between consecutive occurrences of u in w:

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What about MSO$(P_1, P_2, \ldots, P_k)$?

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Much (ongoing) work on this central question! By e.g., Elgot, Rabin, Carton, Thomas, Rabinovich, Fijalkow, Paperman, . . .

Toric words

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Theorem

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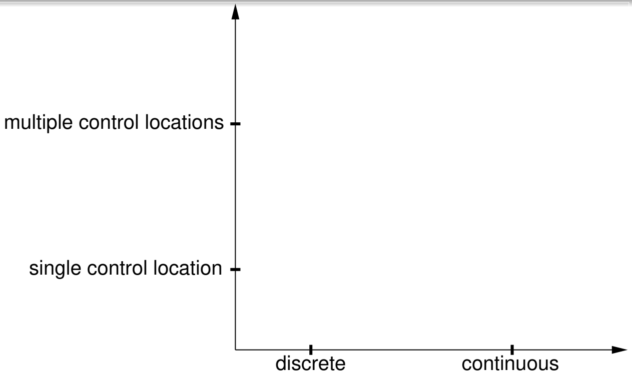
Corollary

Let (M, s) be a linear dynamical system in ambient space $\mathbb{R}^d$, and let $S_1, \ldots, S_k \subseteq \mathbb{R}^d$ be tame semialgebraic predicates.
Let $P_1, \ldots, P_k \subseteq \mathbb{N}$ be the set of visiting times of the orbit of (M, s) in $S_1, \ldots, S_k$ respectively.
Then $MSO(P_1, \ldots, P_k)$ is decidable.

The Algorithmic Theory of Linear Systems



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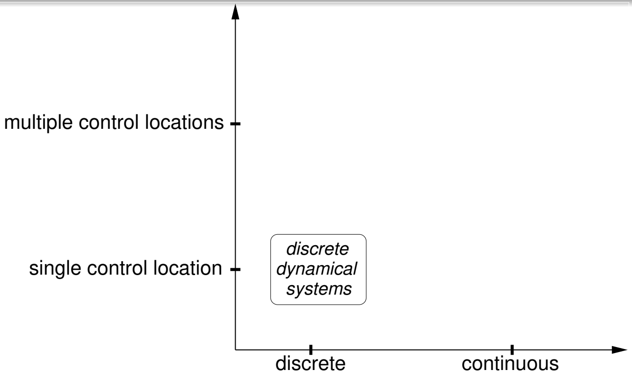

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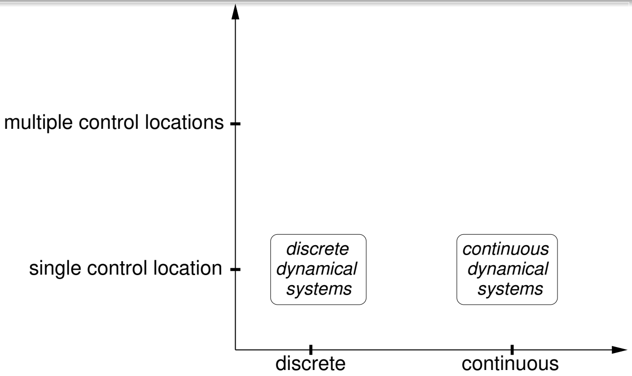

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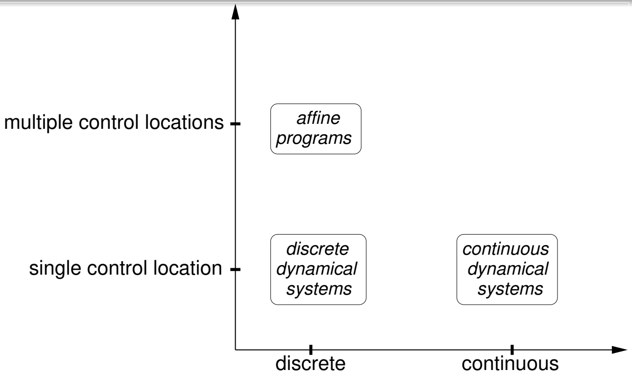

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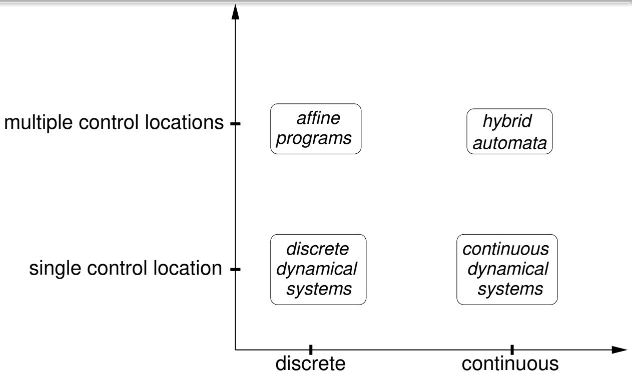

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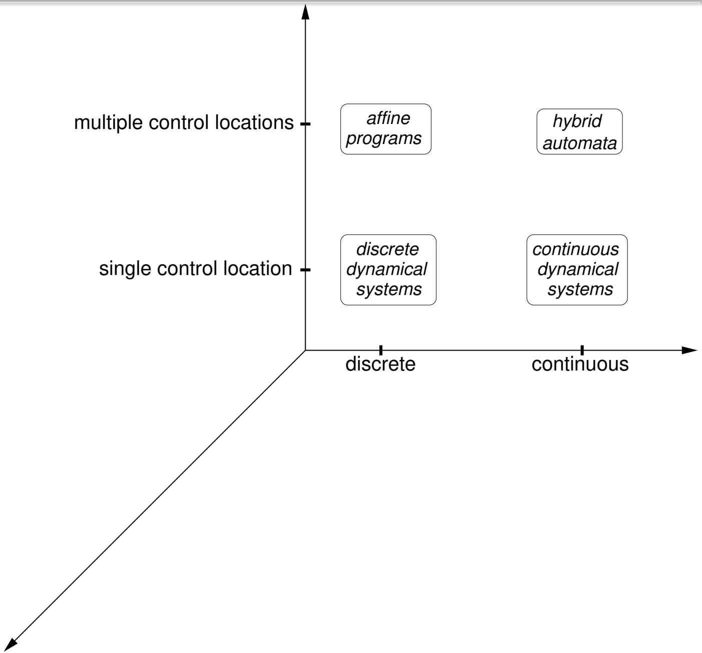

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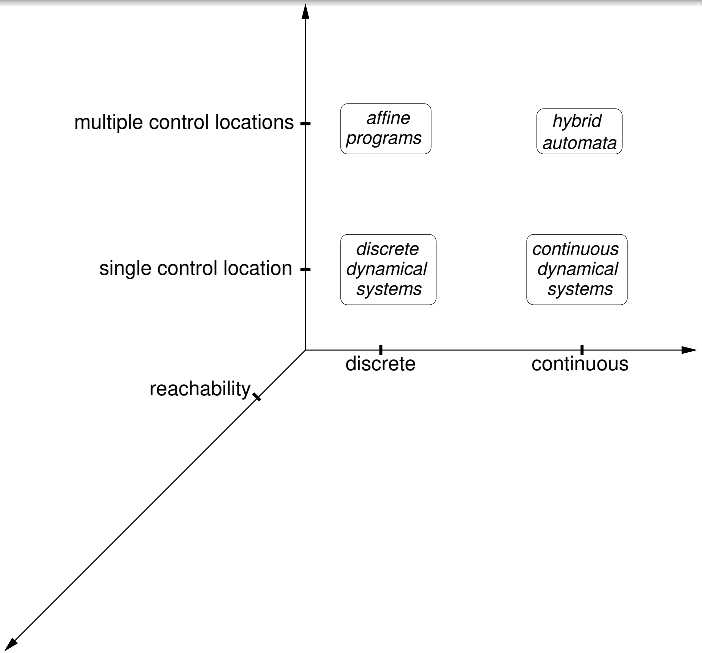

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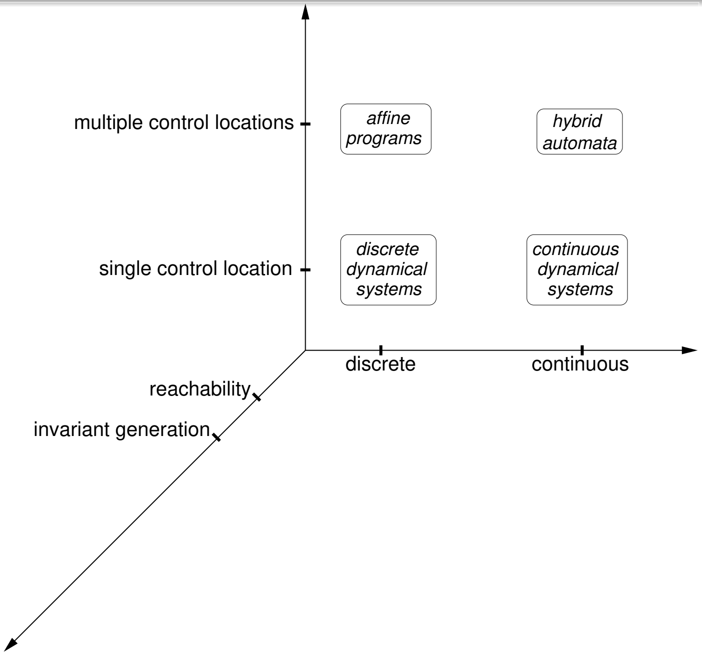

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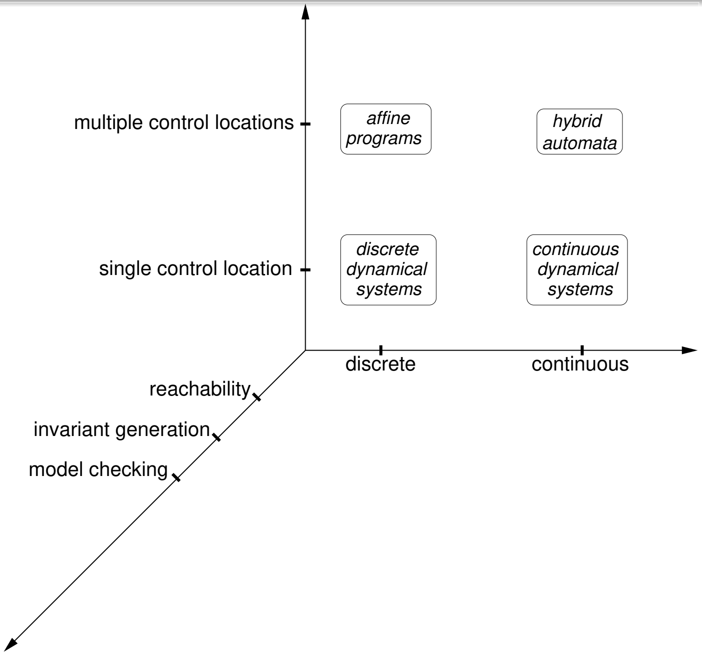

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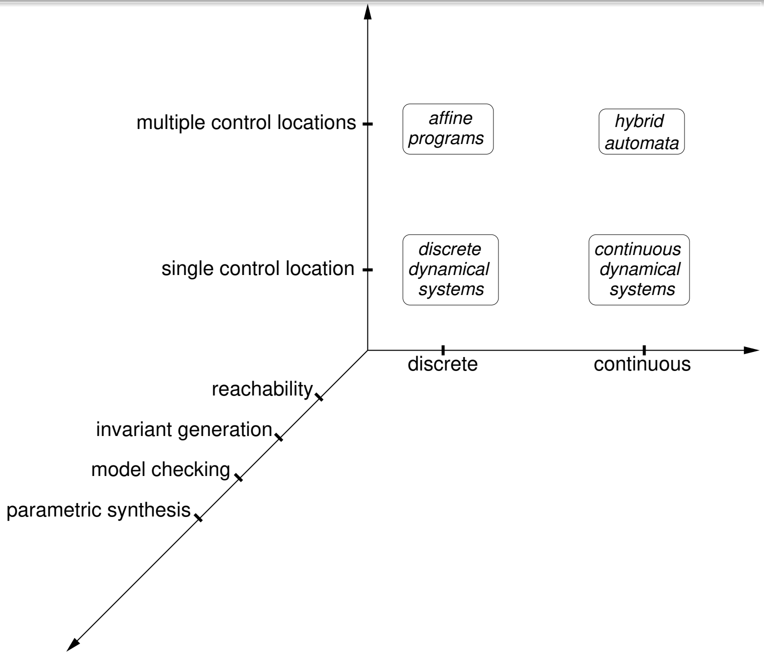

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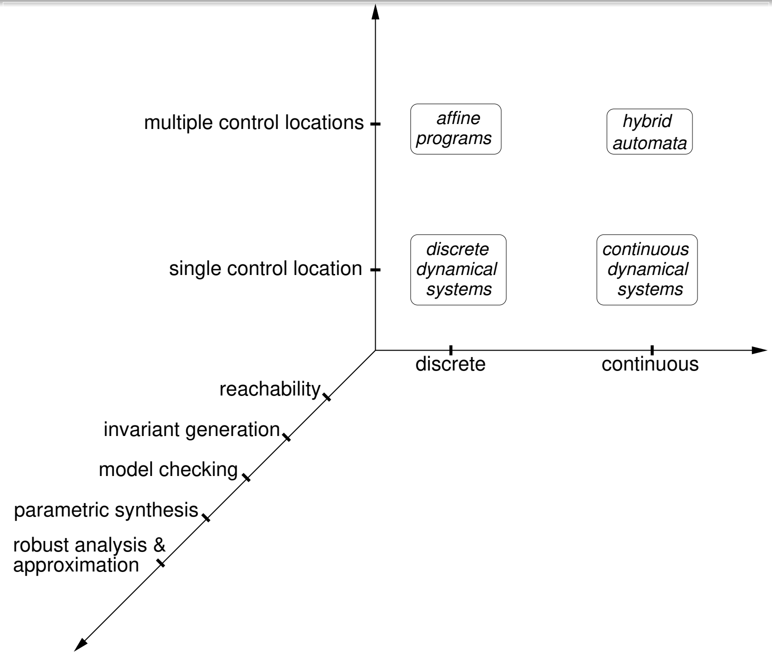

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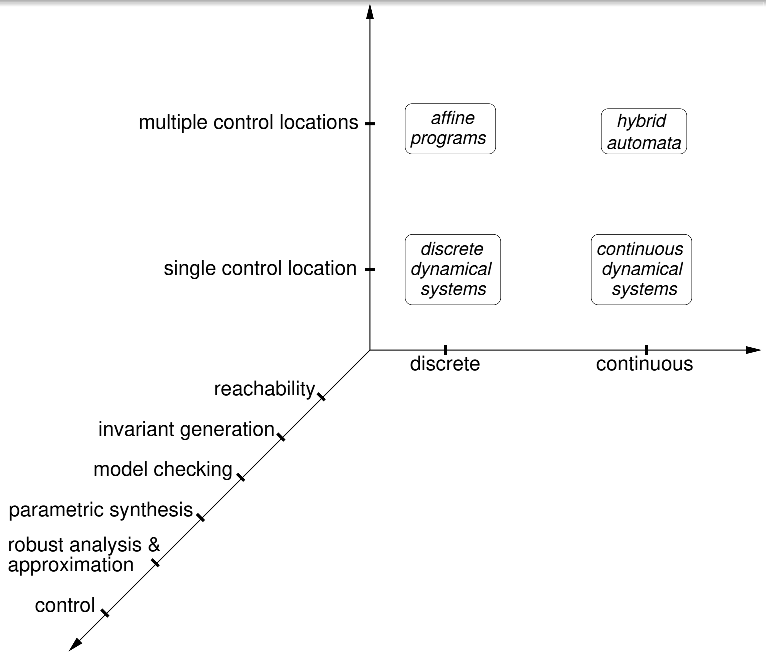

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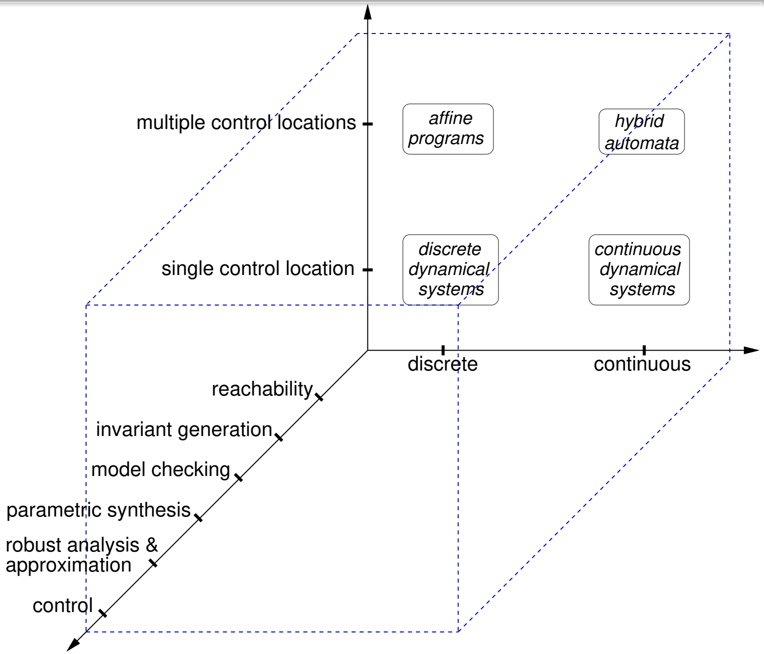

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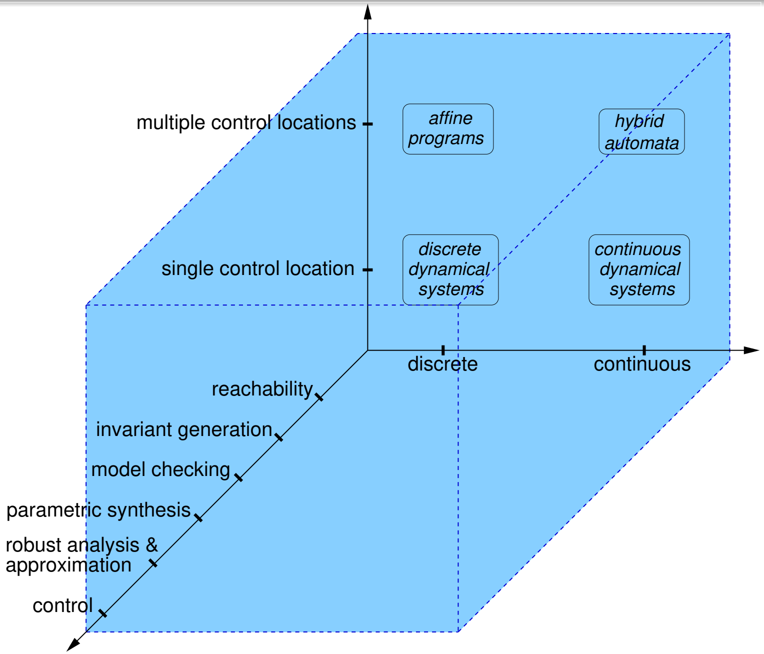

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