Strictly Locally Testable and Resources Restricted Control Languages in Tree-Controlled Grammars

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Theorietag, Kaiserslautern, October 4 – 6, 2023

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- Many papers have been published by different authors on subregular families of languages.
- Focus is often on the decrease of descriptional or computational complexity when going from arbitrary regular languages to special ones.
- Here, the generative capacity of tree-controlled grammars with special regular control languages is considered.

Tree-Controlled Grammars

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- introduced by Karel Čulik II and Hermann A. Maurer in 1977
- formal model for generating languages,
- enhance a context-free grammar by a mechanism to control the derivation,
- words on each level of the derivation tree must belong to a given regular language.

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Search for families X with $CF \subset \mathcal{TC}(X) \subset CS$

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Definitions

Tree-controlled grammar with control in $\mathcal{F}$: $G = (N, T, P, S, R)$ where

- (N, T, P, S) is a context-free grammar with
  - a set N of non-terminal symbols,
  - a set T of terminal symbols,
  - a set P of context-free non-erasing rules (except $S \to \lambda$),
  - an axiom $S \in N$,
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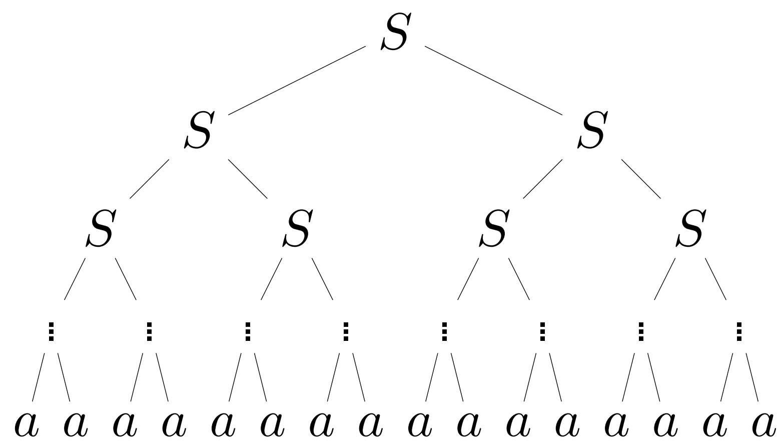
Language family: $\mathcal{TC}(\mathcal{F}) = \{\, L(G) \mid G = (N, T, P, S, R),\ R \in \mathcal{F} \,\}$

Example

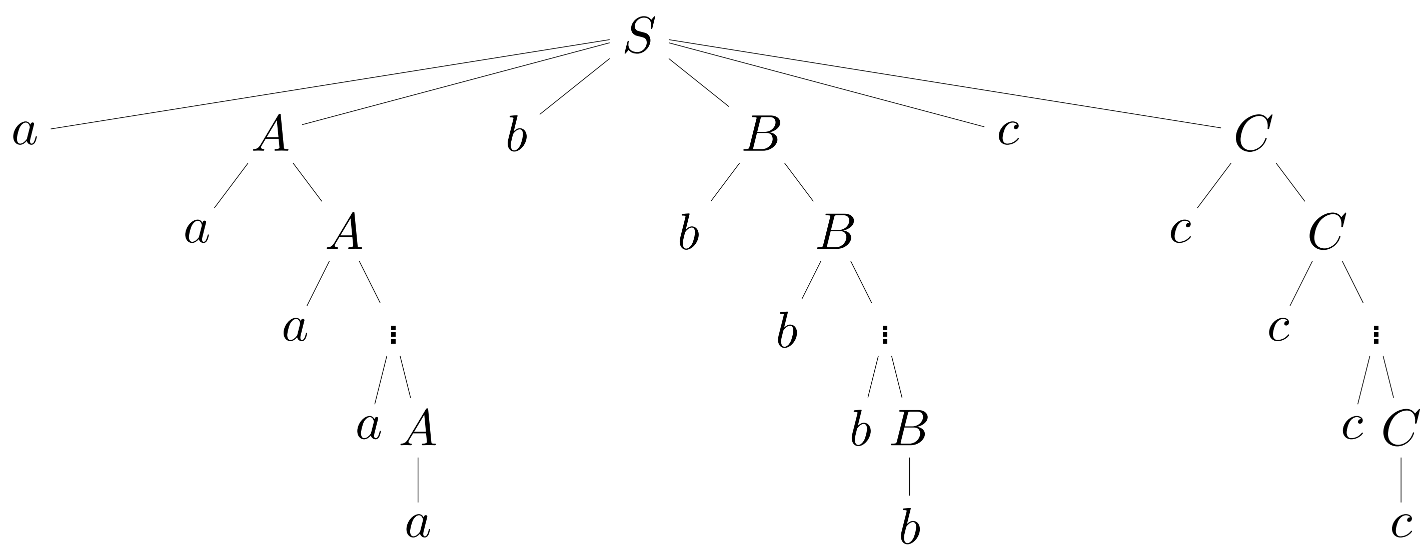
$$G_1 = (\{S\}, \{a\}, \{S \to SS, S \to a\}, S, \{S\}^*)$$



$$\rightsquigarrow L(G_1) = \{\, a^{2^n} \mid n \geq 0 \,\}$$

Another Example

$G_2 = (\{S, A, B, C\}, \{a, b, c\}, P, S, \{S, aAbBcC\})$ with
$P = \{S \to aAbBcC, A \to aA, B \to bB, C \to cC, A \to a, B \to b, C \to c\}$



$\rightsquigarrow L(G_2) = \{ a^n b^n c^n \mid n \geq 2 \}$

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- *suffix-closed* (or *fully initial* or *multiple-entry* language) if and only if, for any words $x \in V^*$ and $y \in V^*$, the relation $xy \in L$ implies $y \in L$,

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- *commutative* if and only if it contains with each word also all permutations of this word,
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- *non-counting* (or *star-free*) if and only if there is a natural number $k \geq 1$ such that, for any words $x \in V^*$, $y \in V^*$, and $z \in V^*$, it holds $xy^kz \in L$ if and only if $xy^{k+1}z \in L$,

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- *union-free* if and only if L can be described by a regular expression which is only built by product and star,
- *strictly locally k-testable* if and only if there are three subsets B, I, and E of V^k such that any word $a_1a_2\ldots a_n$ with $n \geq k$ and $a_i \in V$ for $1 \leq i \leq n$ belongs to the language L if and only if $a_1a_2\ldots a_k \in B$, $a_{j+1}a_{j+2}\ldots a_{j+k} \in I$ $(1 \leq j \leq n-k-1)$, and $a_{n-k+1}a_{n-k+2}\ldots a_n \in E$,

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- *strictly locally testable* if and only if it is strictly locally k-testable for some natural number k.

Further Subregular Language Families

$$REG_n^Z = \{ L \mid L \in REG \text{ with } State(L) \leq n \},$$

where

$$State(L) = \min \{ State(A) \mid A \text{ is a det. finite automaton accepting } L \},$$

with

$$State(A) = |Z|$$

Further Subregular Language Families

$$REG_n^Z = \{ L \mid L \in REG \text{ with } State(L) \leq n \},$$
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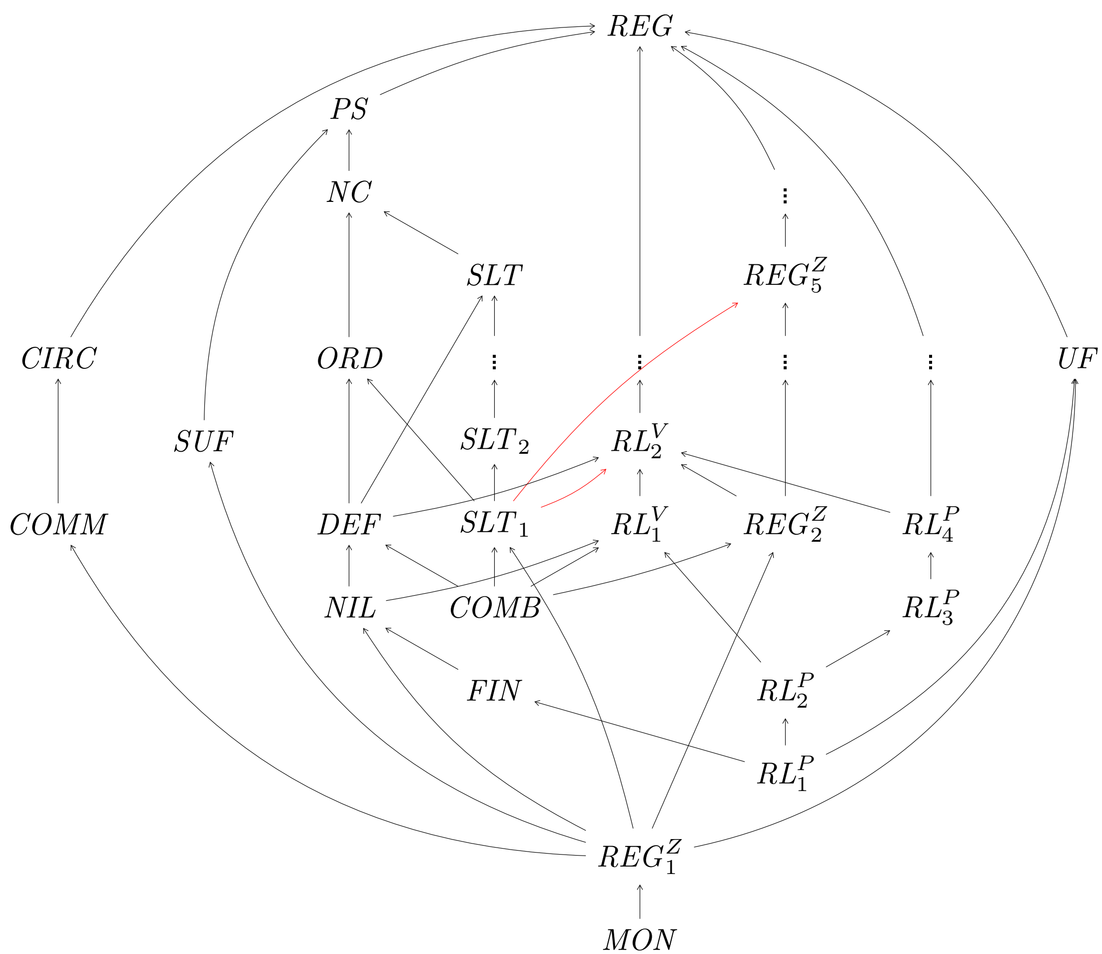
$$State(A) = |Z|, Var(G) = |N|, Prod(G) = |P|.$$

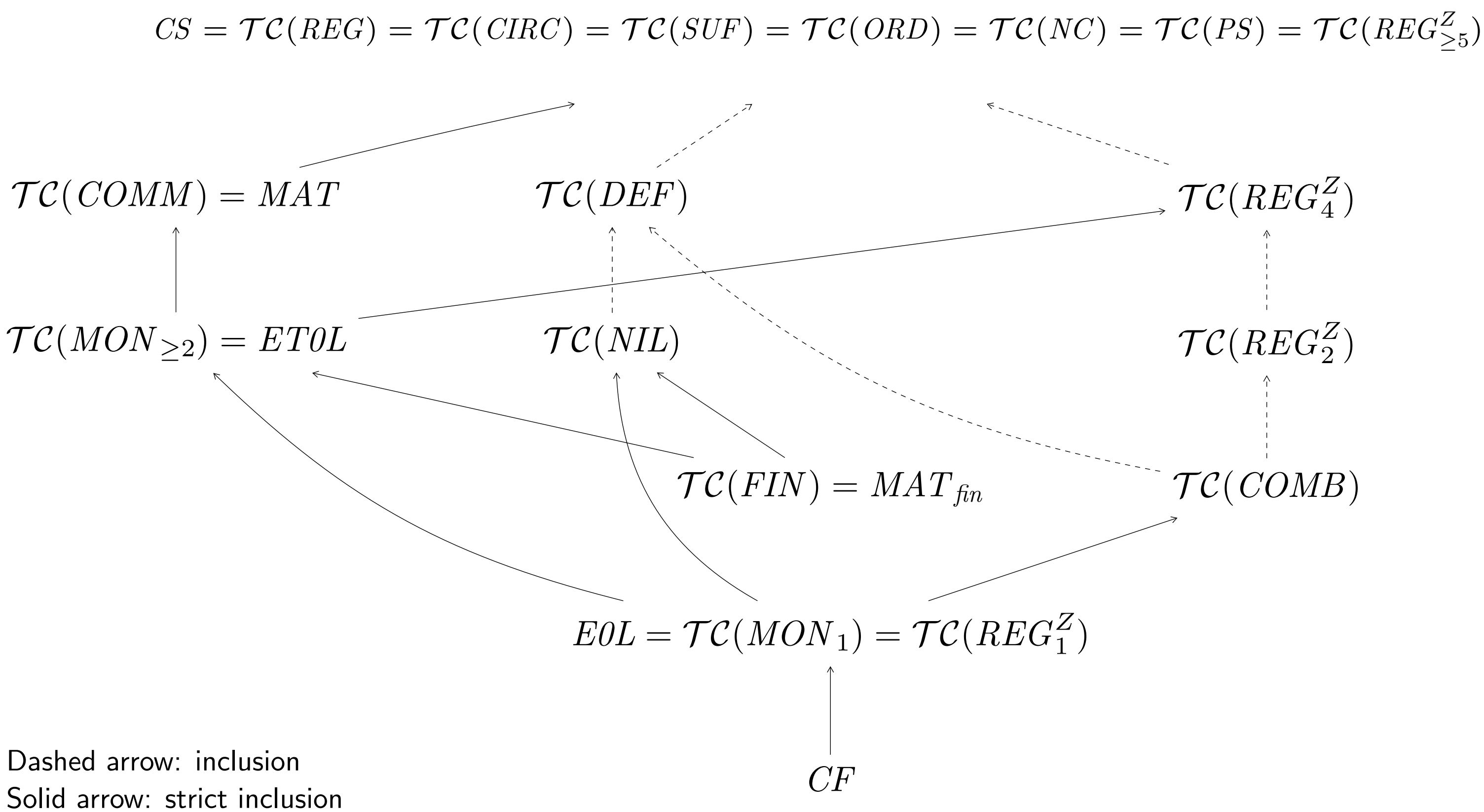
Families Under Consideration

$$\mathcal{F} \in \{FIN, MON, NIL, COMB, DEF, SUF, ORD\}$$
$$\cup \{COMM, CIRC, NC, PS, UF, REG\}$$
$$\cup \{\, SLT_k \mid k \geq 1 \,\} \cup \{SLT\}$$
$$\cup \{\, REG_n^Z \mid n \geq 1 \,\} \cup \{\, RL_n^V \mid n \geq 1 \,\} \cup \{\, RL_n^P \mid n \geq 1 \,\}$$

Hierarchy of Subregular Language Families

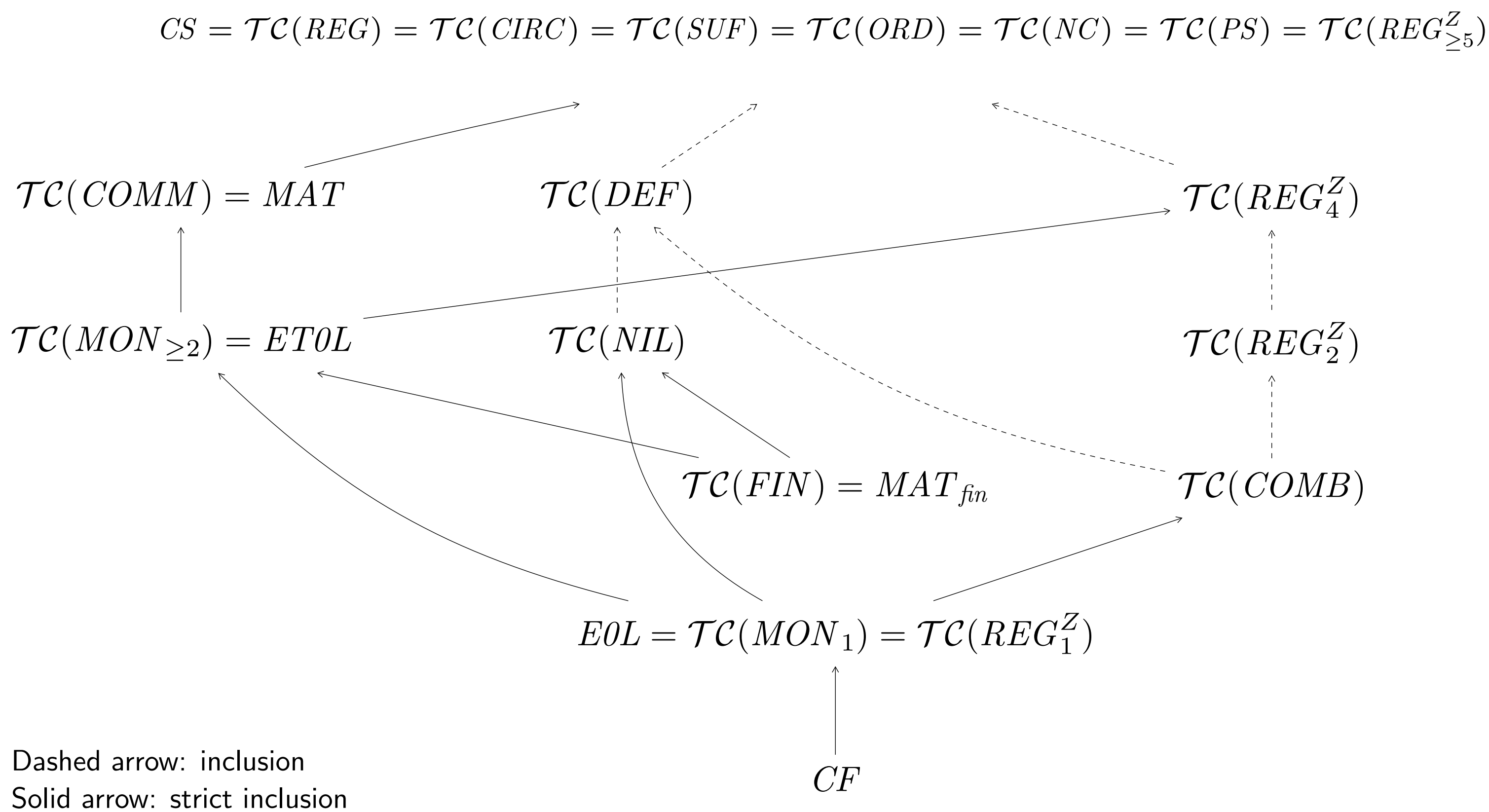


Previous Work



Dashed arrow: inclusion
Solid arrow: strict inclusion

Results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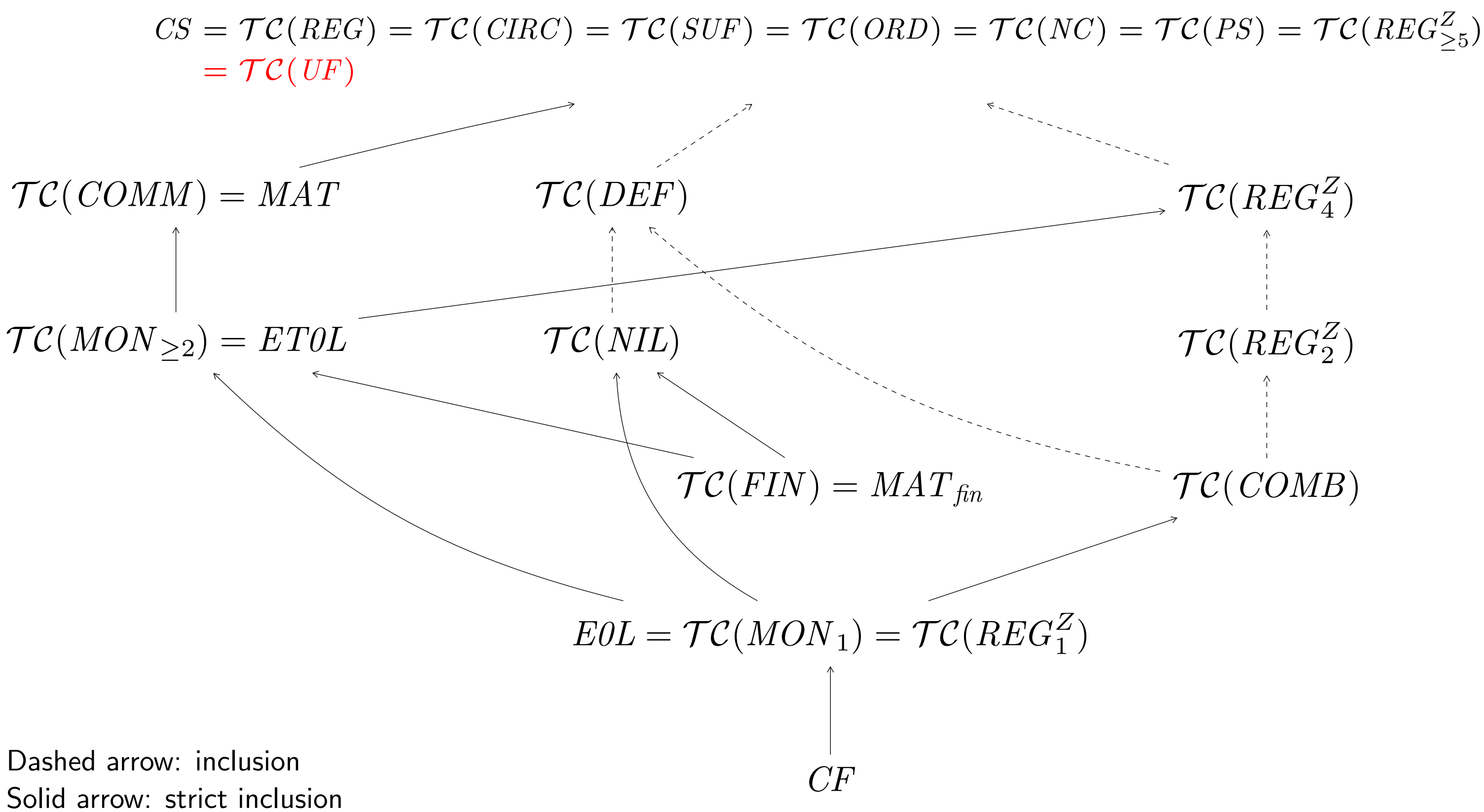
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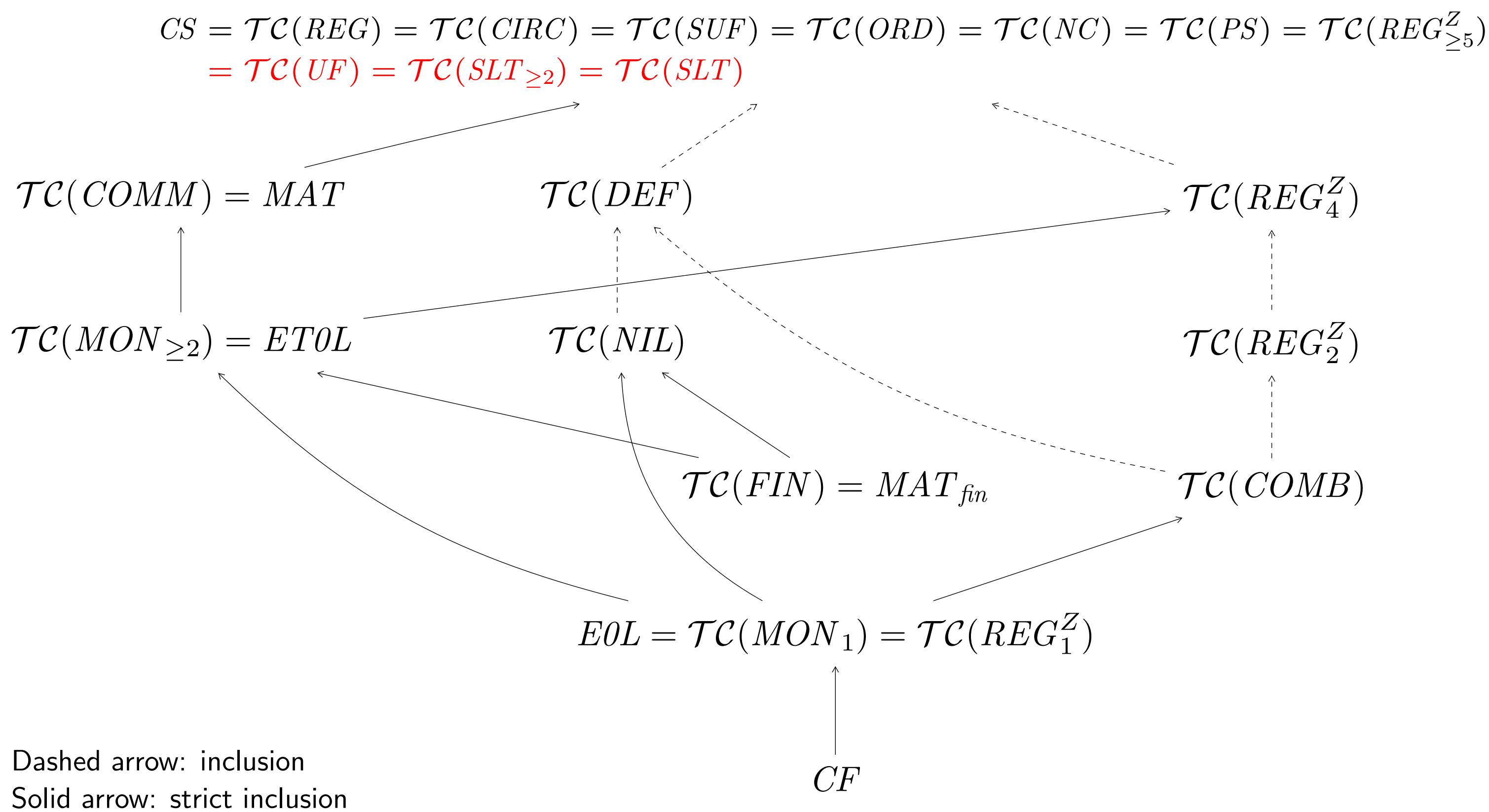

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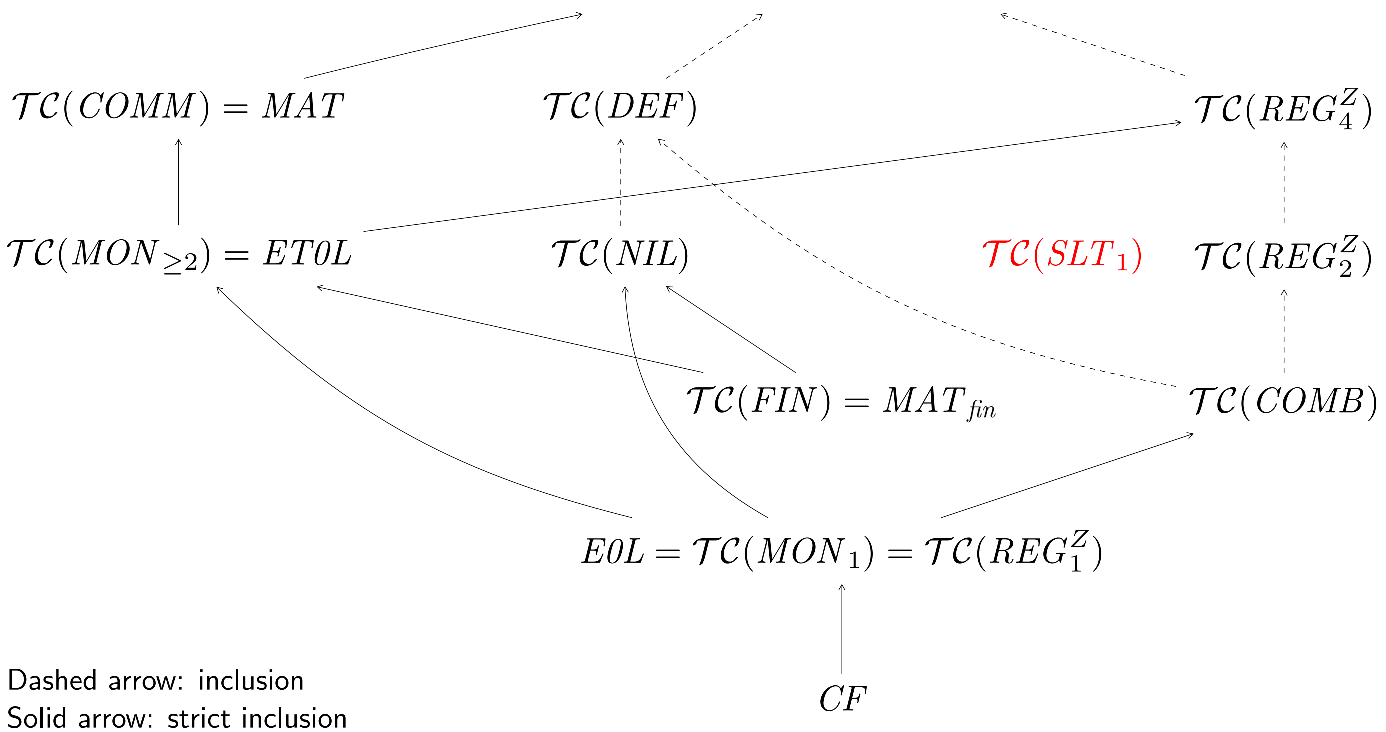

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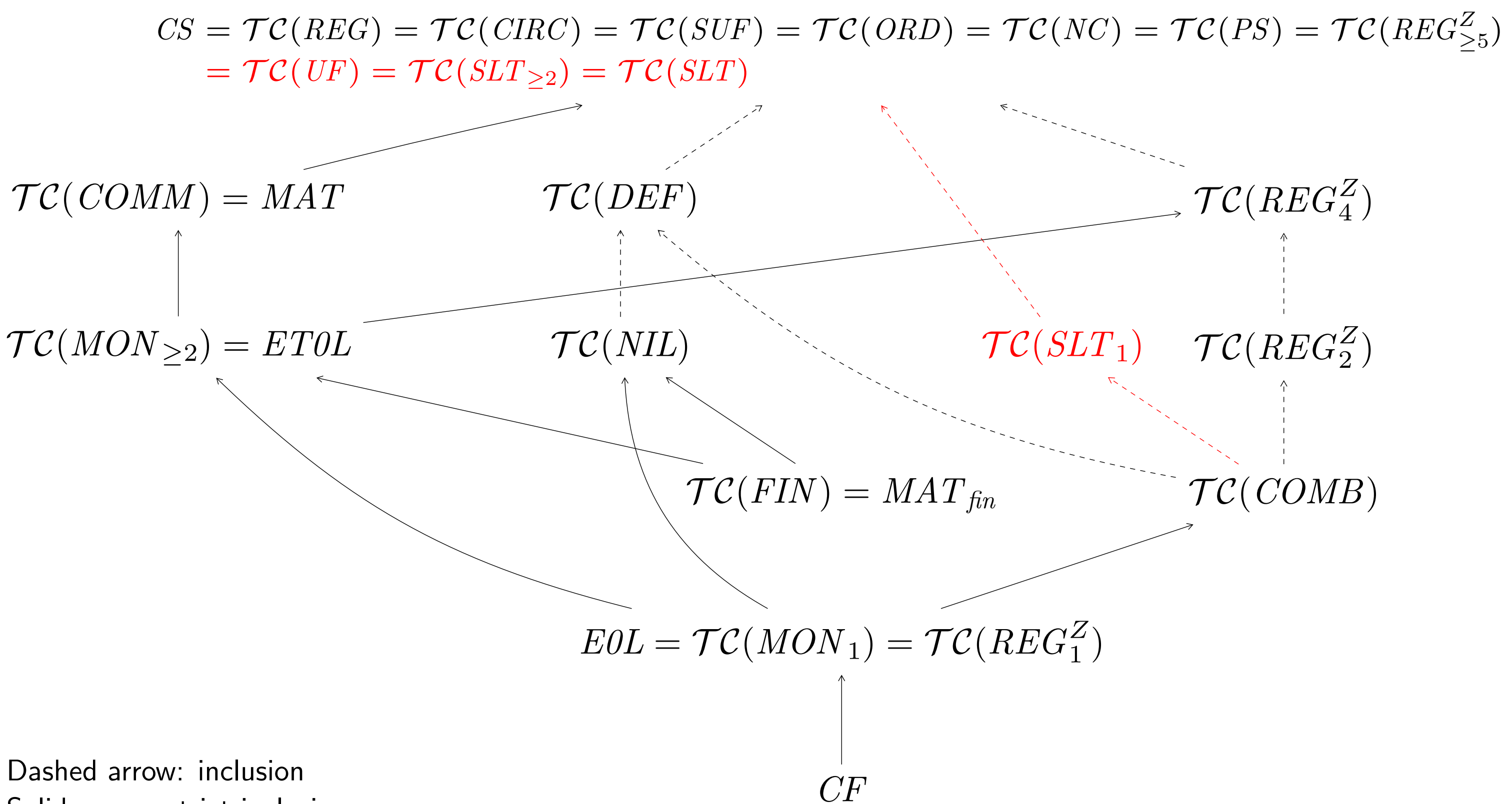

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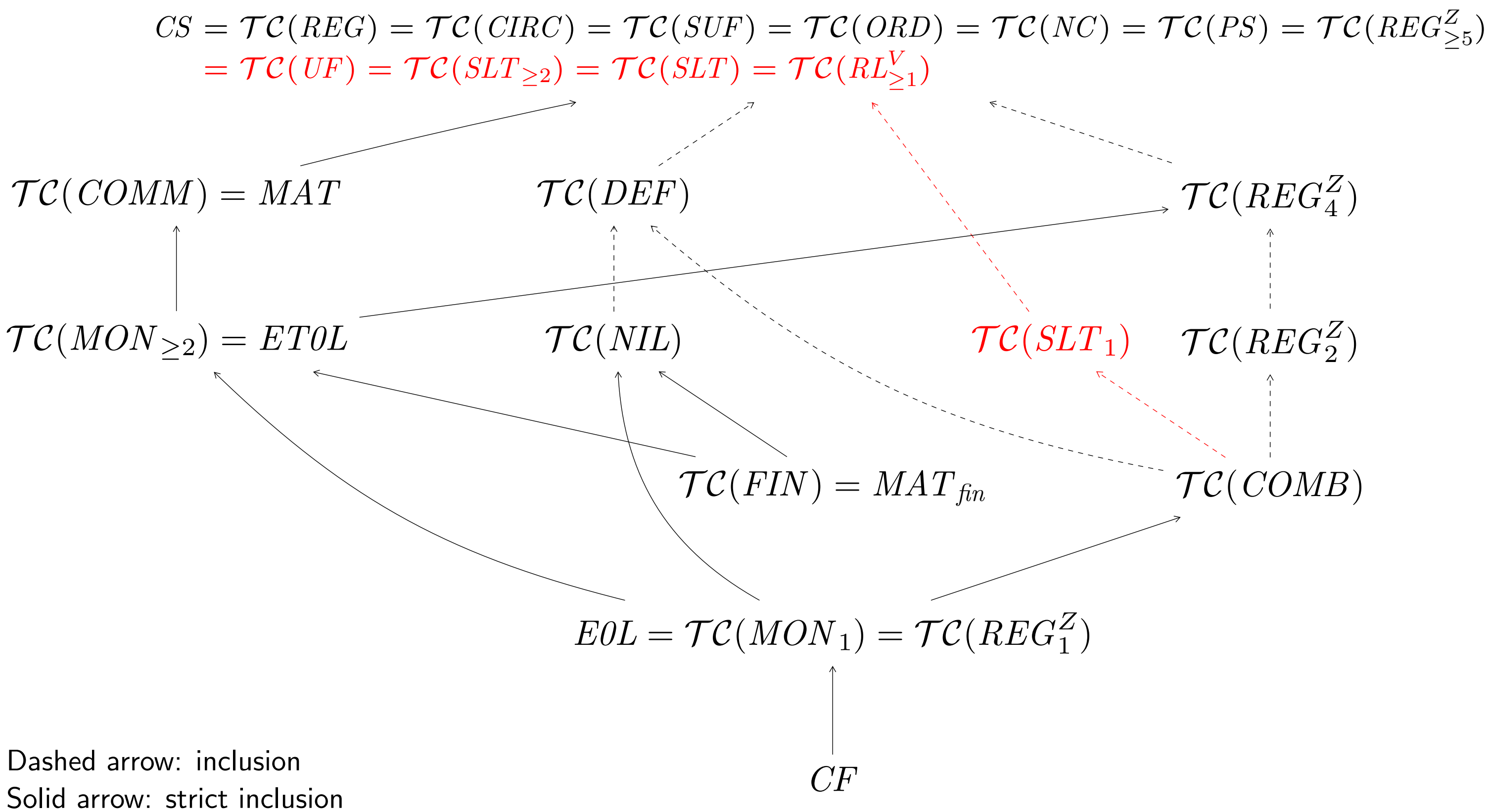
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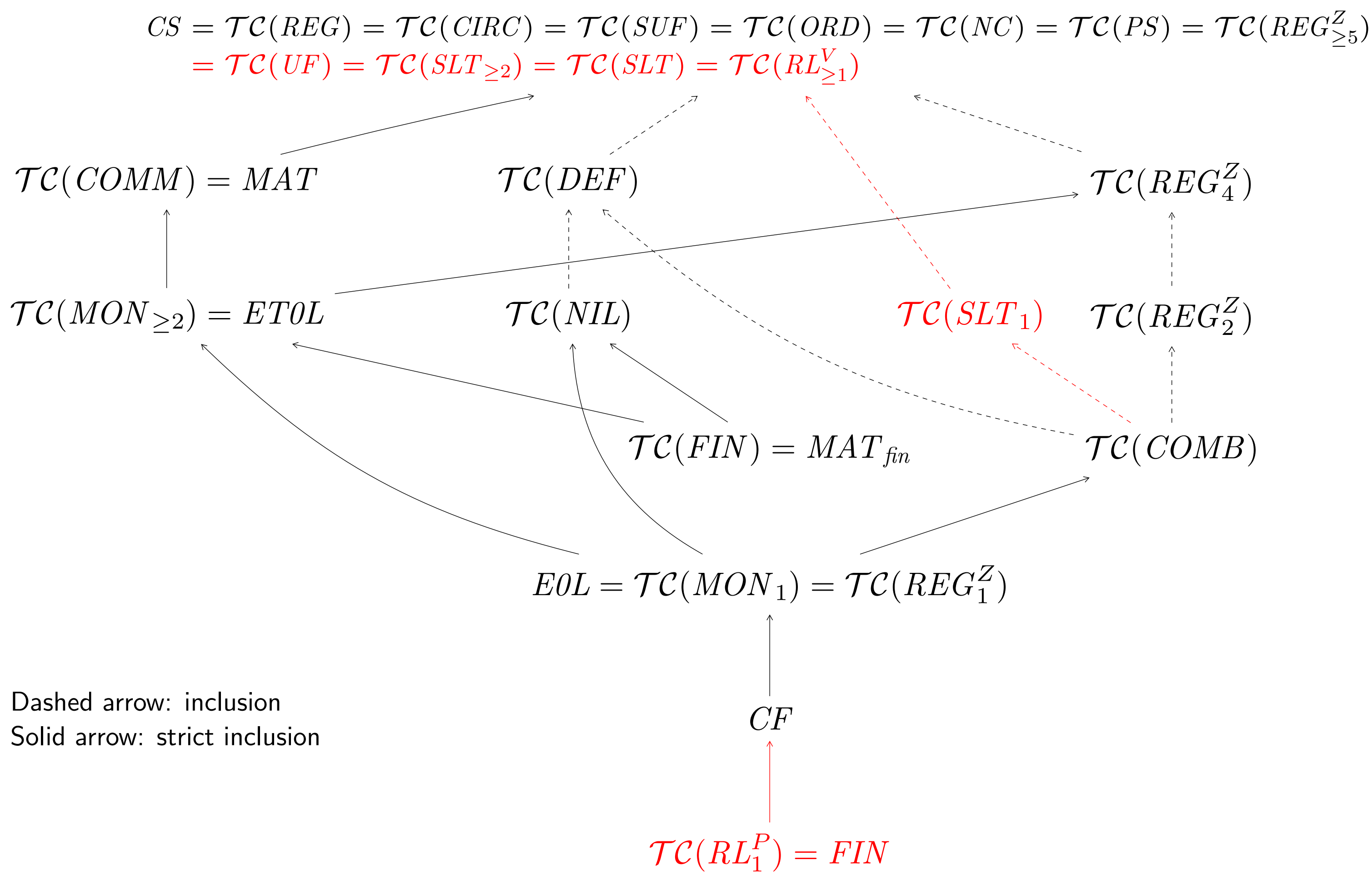
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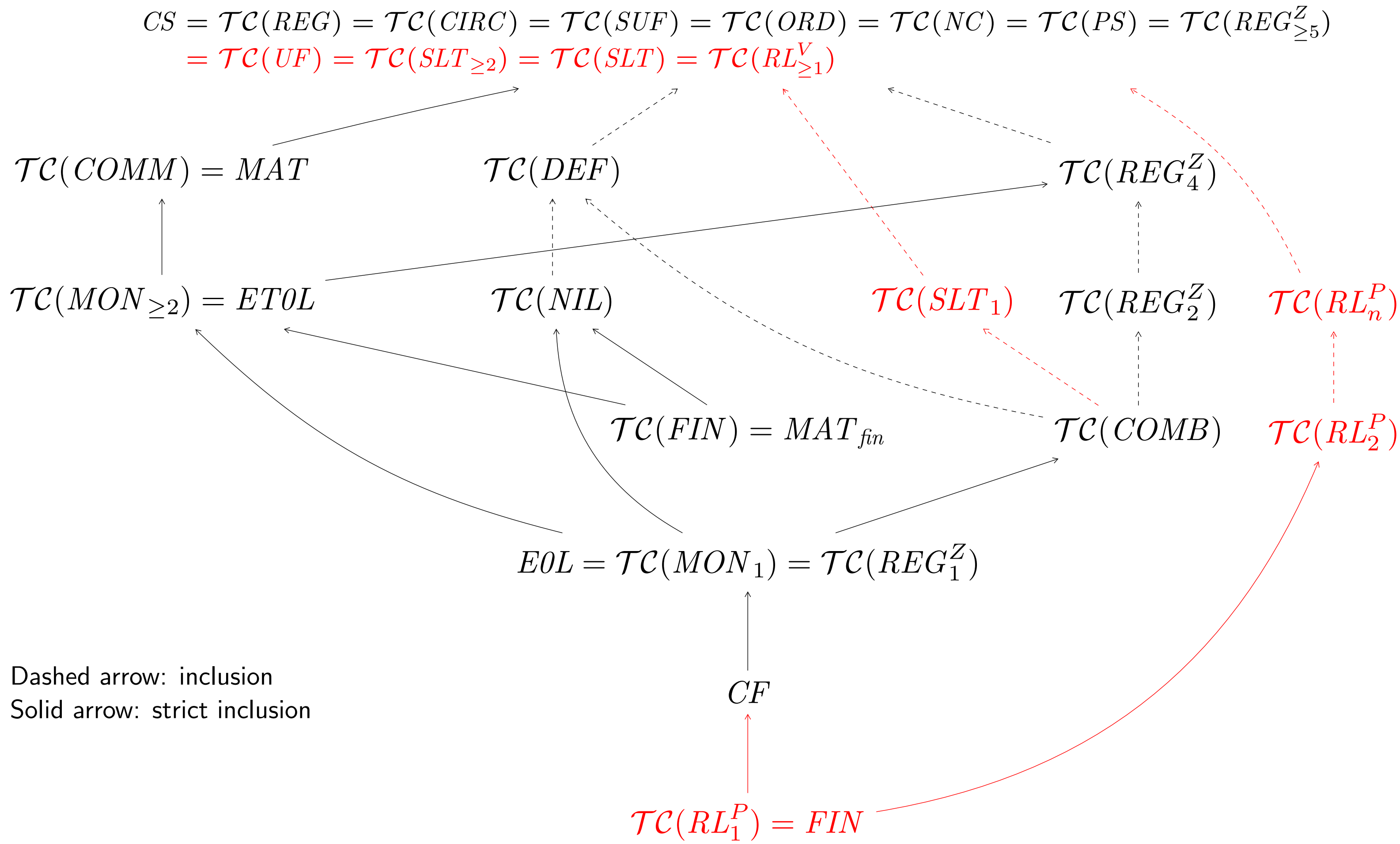
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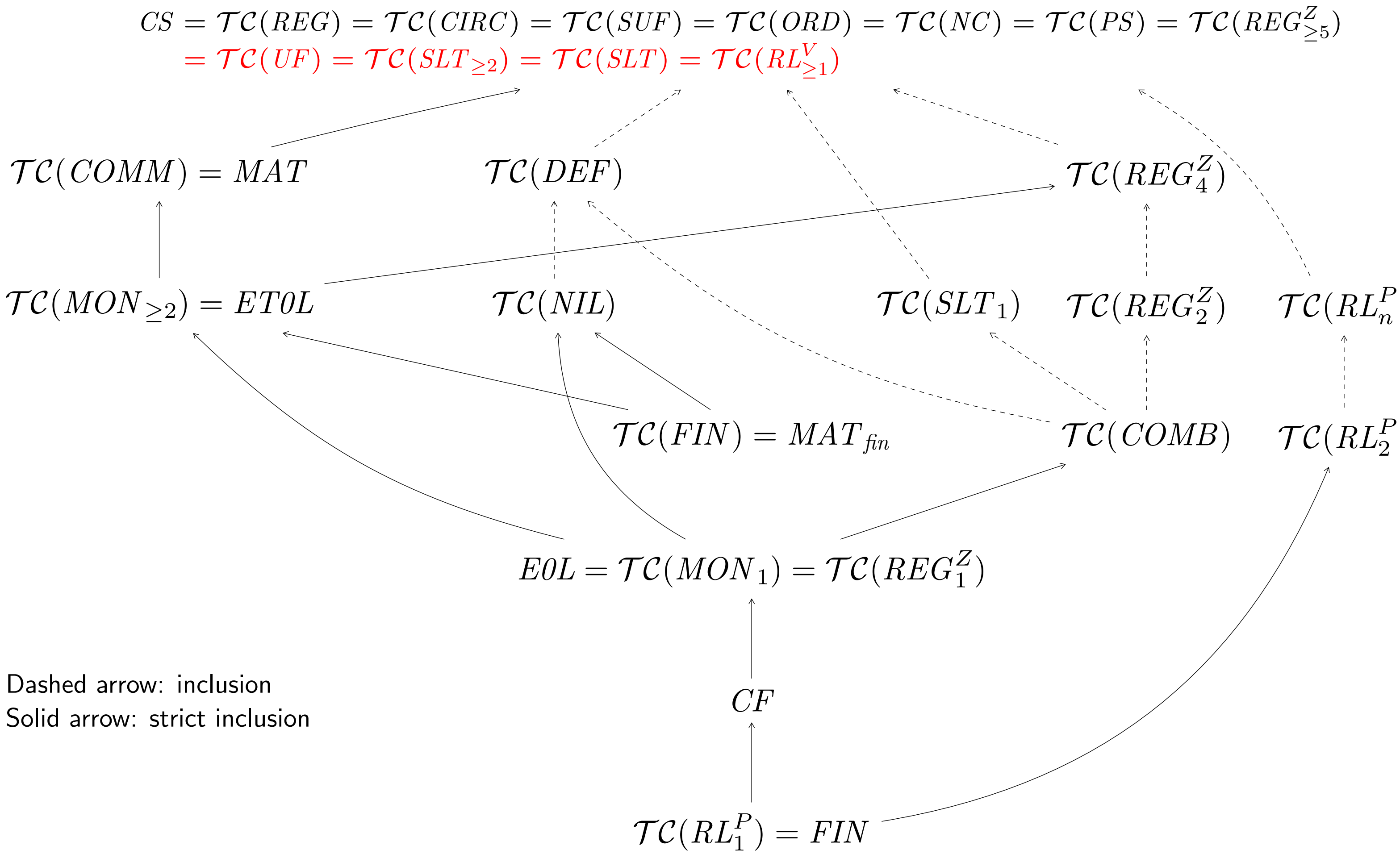
Family of the Context-Sensitive Languages



Simulating a Context-Sensitive Grammar

Kuroda normal form: $G = (N, T, P, S)$ where each rule has the form

- $A \to BC$, $A \to B$, or $A \to a$ with $A, B, C \in N$ and $a \in T$, or
- $AB \to CD$ with $A, B, C, D \in N$

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$p : AB \to CD$ will be simulated by the rules

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Rules of the form $A \to BC$ or $A \to B$ can be simulated by themselves.

Rules of the form $A \to a$ should be modified to $A \to \hat{a}$ with a non-terminal placeholder $\hat{a}$.

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$G = (N, T, P, S)$ as above $\rightsquigarrow$ tc-grammar $G_{\mathrm{tc}} = (N_{\mathrm{tc}}, T, P_{\mathrm{tc}}, S, R_{\mathrm{tc}})$:

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Properties of the Control Language

$R_{\mathrm{tc}} = (N_{\mathrm{cf}} \cup N_{12})^*$ with

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Strictly locally 2-testable: $R_{\mathrm{tc}} = [B, I, E, F]$ with

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Thus, $\mathcal{TC}(SLT_k) = CS$ for $k \geq 2$ and $\mathcal{TC}(SLT) = CS$.

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$R_{\mathrm{tc}} \in UF$: for a finite language $L = \{w_1, w_2, \ldots, w_n\}$, the Kleene-closure is $L^* = (\{w_1\}^*\{w_2\}^* \cdots \{w_n\}^*)^*$ and therefore union-free.

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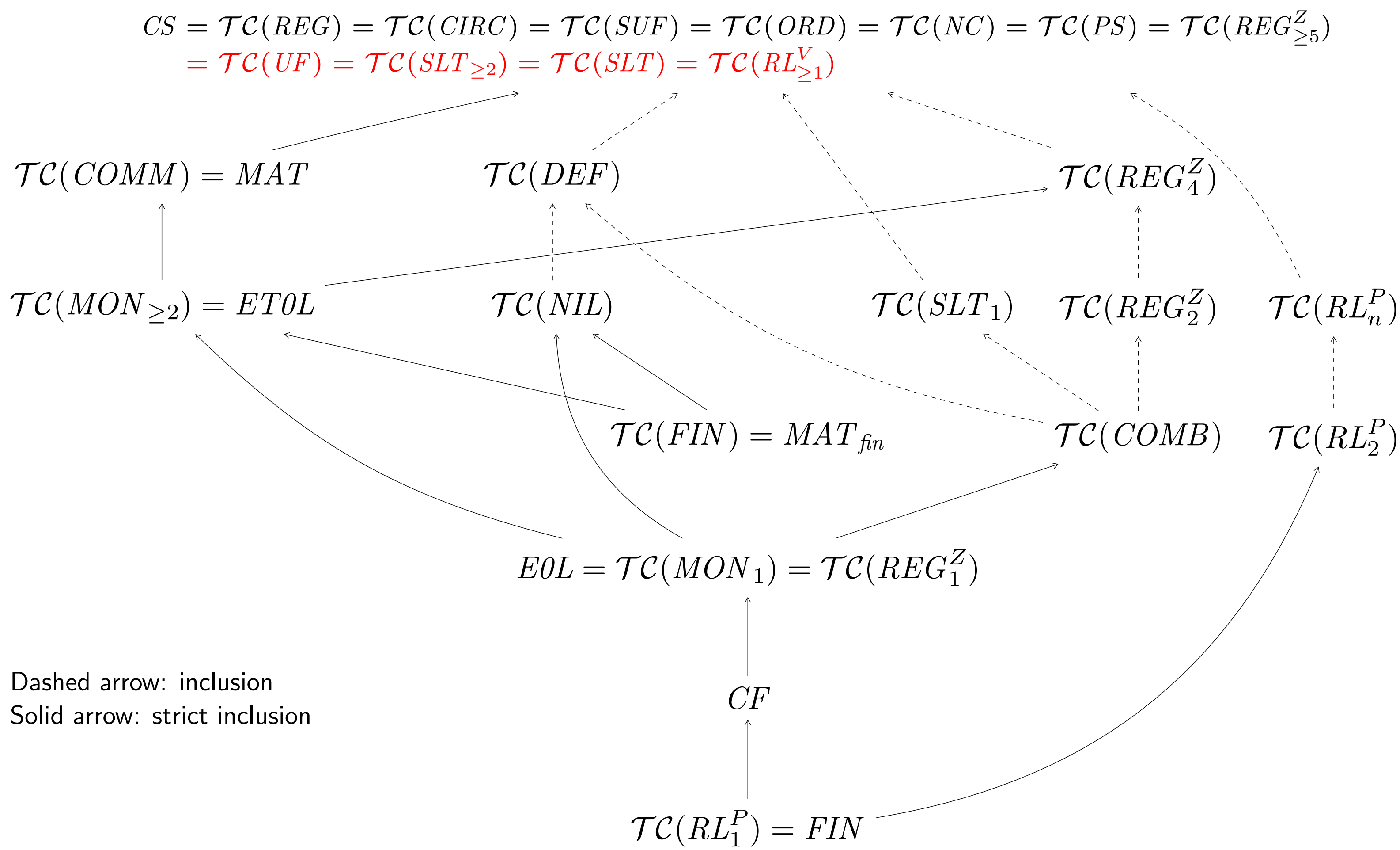
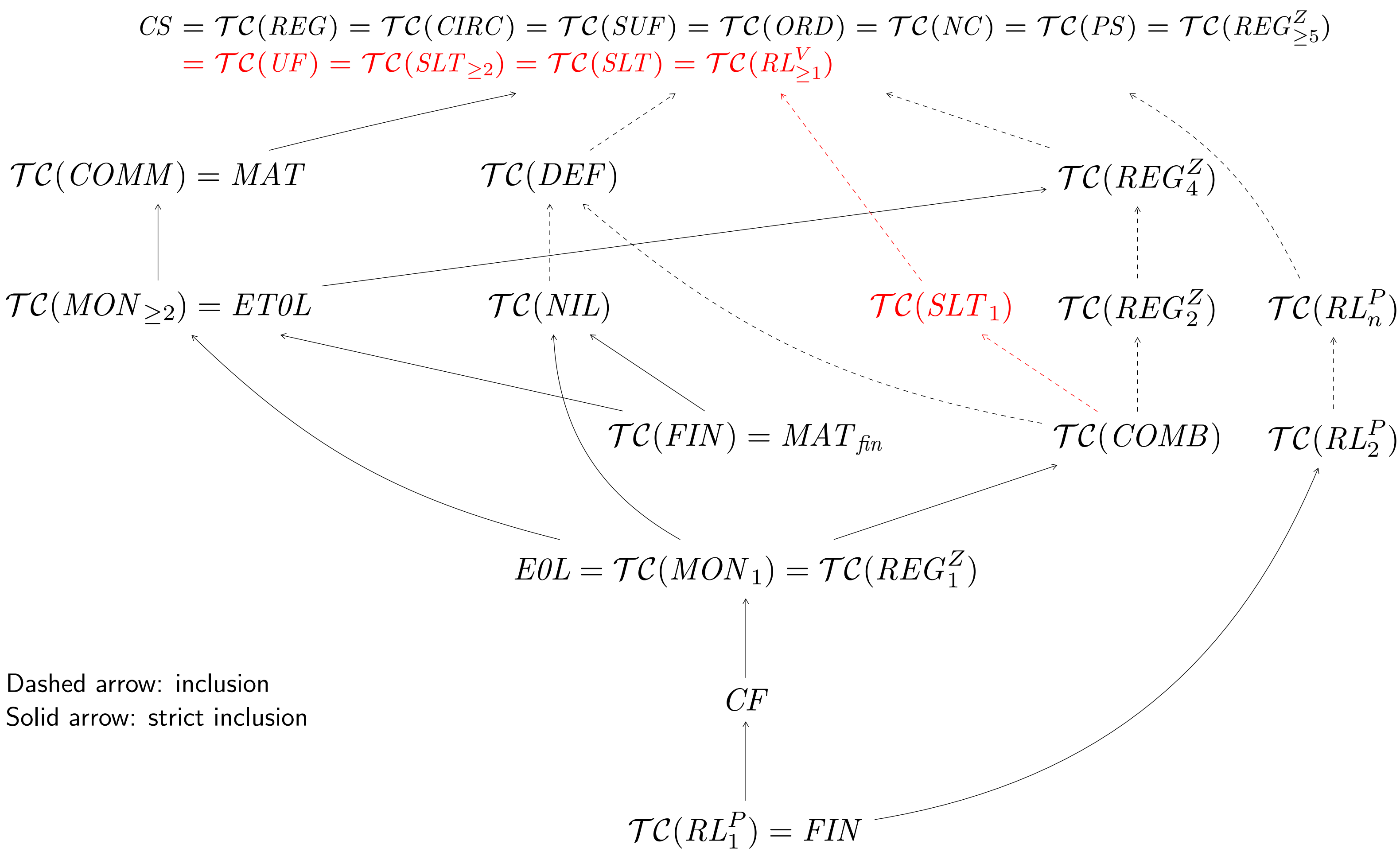
Thus, $\mathcal{TC}(UF) = CS$.

Family of the Context-Sensitive Languages



Family of the Context-Sensitive Languages



Dashed arrow: inclusion
Solid arrow: strict inclusion

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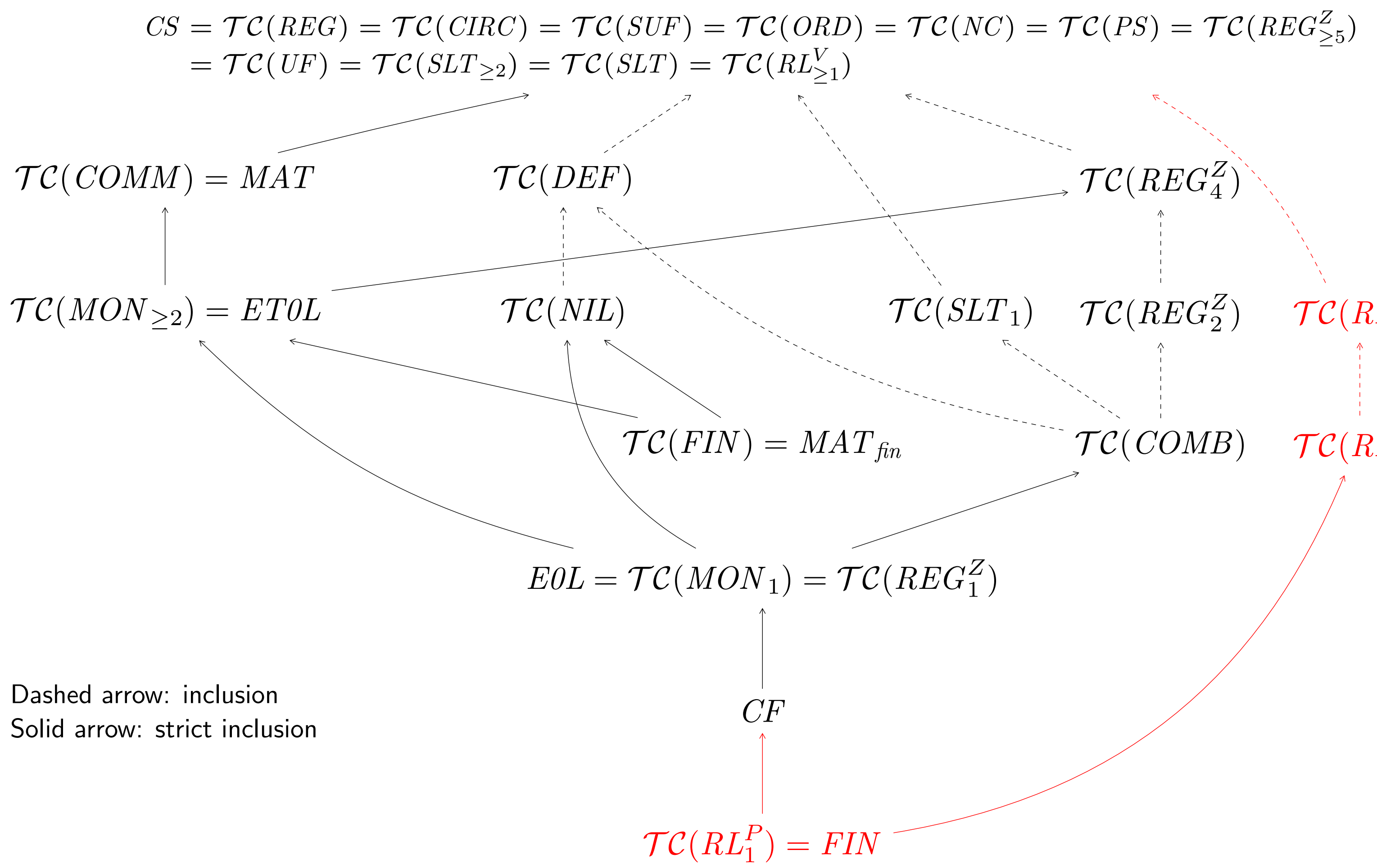

Control Languages generated by one Rule

Let $G = (N, T, P, S, R)$ be a tc-grammar with $R \in RL_1^P$.

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$\rightsquigarrow |R| < 1,$

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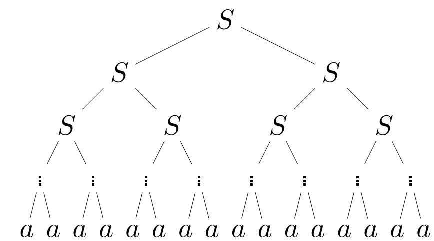
$\rightsquigarrow |L(G)| < \infty$.

Thus, $\mathcal{TC}(RL_1^P) = FIN \subset CF$.

Control Languages generated by two Rules

Examples from the beginning:



$$G_1 = (\{S\}, \{a\}, \{S \to SS, S \to a\}, S, \{S\}^*)$$

$$\rightsquigarrow L(G_1) = \{\, a^{2^n} \mid n \geq 0 \,\}$$

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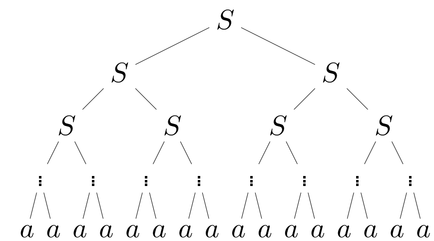

$G_2 = (\{S, A, B, C\}, \{a, b, c\}, P, S, \{S, aAbBcC\})$ with
$P = \{S \to aAbBcC, A \to aA, B \to bB, C \to cC, A \to a, B \to b, C \to c\}$

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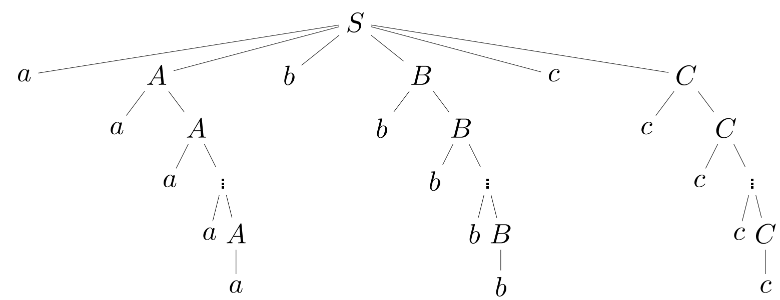

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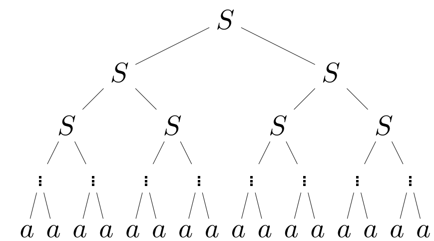

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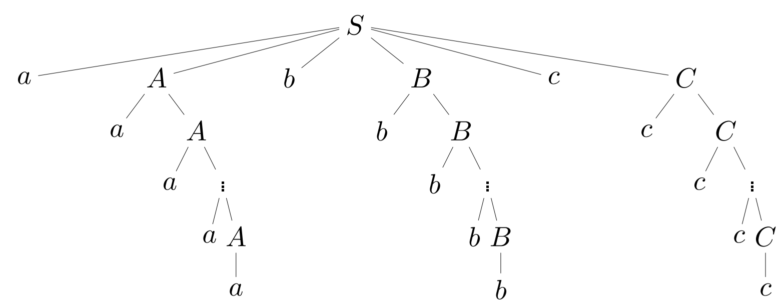

$\rightsquigarrow L(G_i) \in \mathcal{TC}(RL_2^P) \setminus CF$ for $i \in \{1, 2\}$
Thus, $\mathcal{TC}(RL_1^P) \subset \mathcal{TC}(RL_2^P)$.

Future Work

- In many cases, the strictness of the inclusion remains open.
- The incomparability of families is open as well in many cases.
- Consider tree-controlled grammars with erasing rules.
- Consider other subregular control languages.
- Relate the families of languages generated by tree-controlled grammars to language families obtained by other grammars/systems with regulated rewriting.