Regular Separability in Büchi VASS

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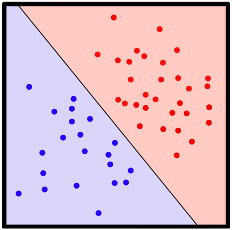
Separators are Useful



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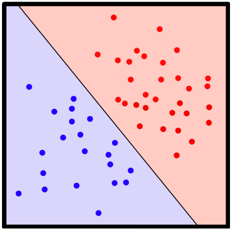

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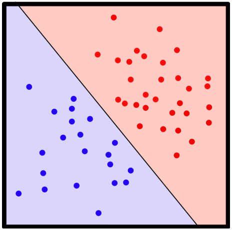

- Many applications to finding separators for two sets of data.
- For formal languages, separate good and bad execution traces.
- Can answer safety/reachability questions, serve as over-approximation.

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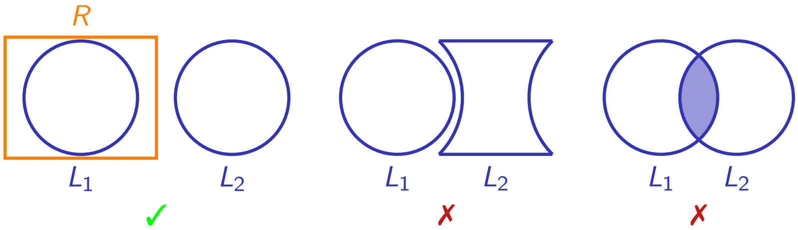

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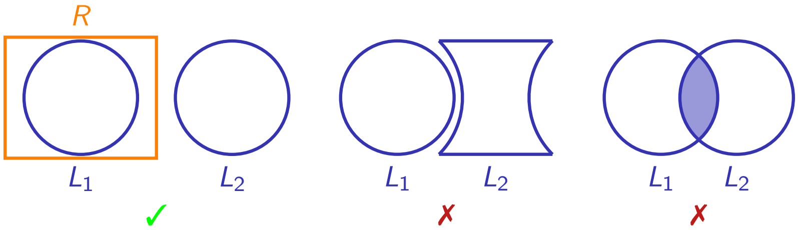

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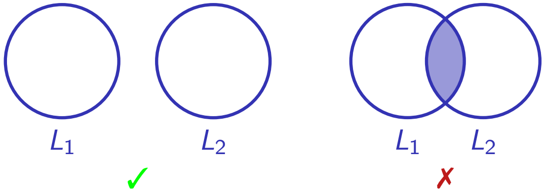

Regular Separability ≡ Disjointness in Coverability VASS

Theorem [Czerwinski, Lasota, M., Muskalla, Kumar, Saivasan, 2018]

Let L_1, L_2 be languages of coverability VASS. If $L_1 \cap L_2 = \emptyset$, then one can always find a regular separator R with $L_1 \subseteq R$ and $L_2 \cap R = \emptyset$.

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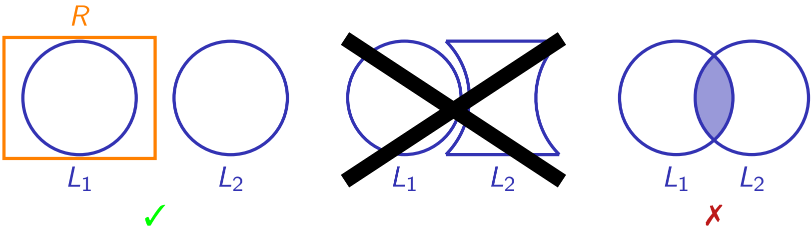

Infinite Traces: Separators are Still Useful

Consider formal languages over infinitely long words.

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- Instead of safety, can answer liveness questions.
- For VASS we naturally arrive at the Büchi acceptance condition.

Remainder of the Talk

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- Show that regular separability $\not\equiv$ disjointness in Büchi VASS.
- Present the ideas needed to prove our main result:

Theorem

Regular separability in Büchi VASS is decidable.

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A language is ω-regular iff it is accepted by a Büchi automaton.

- I.e. a finite state automaton with the Büchi acceptance condition.

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Input Two Büchi VASS $\mathcal{V}_1$, $\mathcal{V}_2$.

Question Is there an ω-regular R with $L(\mathcal{V}_1) \subseteq R$ and $L(\mathcal{V}_2) \cap R = \emptyset$?

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We also write $L(\mathcal{V}_1) \mid L(\mathcal{V}_2)$ to denote regular separability.

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Balance: $\varphi_1 : \Sigma^* \to \mathbb{Z}, w \mapsto |w|_a - |w|_{\bar{a}}$. //How many more as than $\bar{a}$s.

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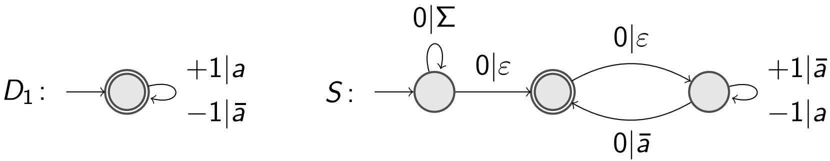

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This is because S contains $w_n := (a^{n+1}\bar{a}^{n+2})^\omega$ for all $n \in \mathbb{N}$.

For an n-state Büchi automaton with an accepting run for w_n, we can pump each block a^{n+1}, which yields an accepting run for a word in D_1.

Talk and Proof Outline

- ✓ Formally define Büchi VASS and their regular separability problem.
- ✓ Show that regular separability $\neq$ disjointness in Büchi VASS.

- Prove decidability of regular separability in Büchi VASS.
  - Wlog. one input is the generator language D_n.
  - How do possible separators look like in this case?
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Lemma (fixing one input as D_n)

Given $\mathcal{V}_1$ and $\mathcal{V}_2$, we can compute in polynomial-time a Büchi VASS $\mathcal{V}$ so that $L(\mathcal{V}_1) \mid L(\mathcal{V}_2)$ if and only if $L(\mathcal{V}) \mid D_n$.

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The proof is similar to the finite word case (Czerwinski, Z., 2020).

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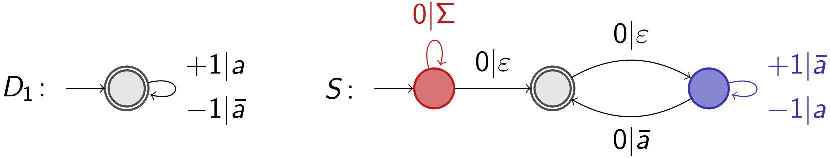

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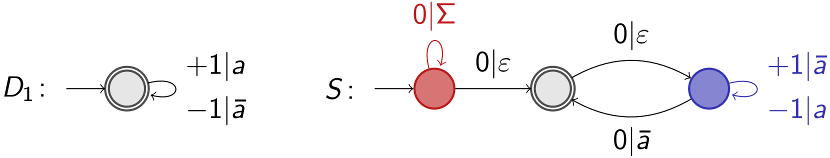

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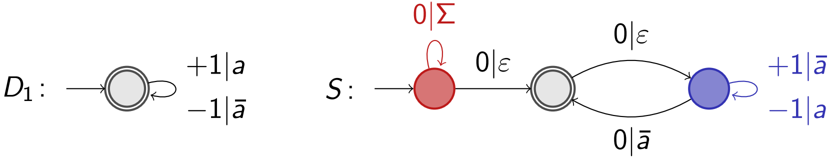

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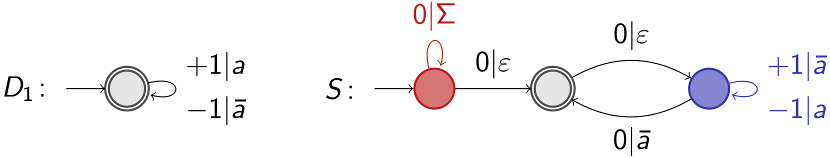

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Theorem (basic separators)

Let $R \subseteq \Sigma_n^\omega$ be ω-regular with $R \cap D_n = \emptyset$. Then R is included in a finite union of languages $P_{i,k}$ and $S_{\boldsymbol{x},k}$.

Proving Basic Separators

Assume ω-regular R is disjoint from D_n.
Construct a system of equations $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ based on R's automaton.

Farkas' Lemma

Let $\boldsymbol{A} \in \mathbb{Q}^{m \times n}$ be a matrix and let $\boldsymbol{b} \in \mathbb{Q}^m$ be a column vector. Then the system $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ either has a solution $\boldsymbol{x} \in \mathbb{Q}^n_{\geq 0}$, or there is a vector $\boldsymbol{y} \in \mathbb{Q}^m_{\geq 0}$ with $\boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{0}$ and $\boldsymbol{y}^\top \boldsymbol{b} < 0$.

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If we obtain a vector $\boldsymbol{y}$, then this implies $R \cap D_n \neq \emptyset$.

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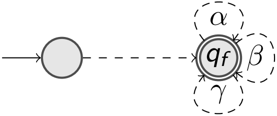

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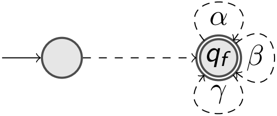

subject to the following conditions:

1. The combined cycle $\alpha\beta\gamma$ adds a vector in $\mathbb{N}^d$ in the VASS.
2. The combined cycle $\alpha\beta$ has non-negative letter balance: $\varphi(\alpha\beta) \geq \mathbf{0}$.
3. Cycles $\alpha\beta\gamma$ and α have the following relationship: $\varphi(\alpha\beta\gamma) \in \mathbb{Q} \cdot \varphi(\alpha)$.

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Results

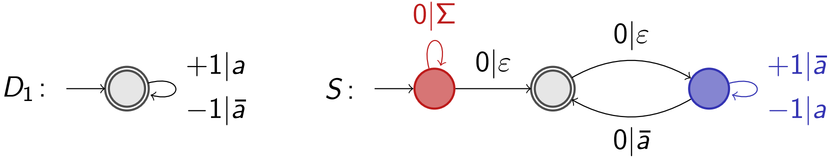
Model	Lower Bound	Upper Bound
1-dim. Büchi VASS (binary enc.)	PSPACE	PSPACE
Arbitrary Büchi VASS	EXPSPACE	ACKERMANN

Lower bounds follow from disjointness.

PSPACE upper bound follows from config. reachability in 2-dim. VASS.

ACKERMANN upper bound follows from size of the Karp-Miller graph.

Conclusion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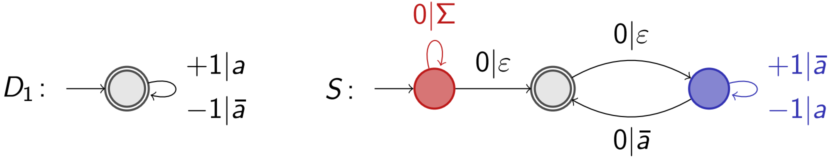

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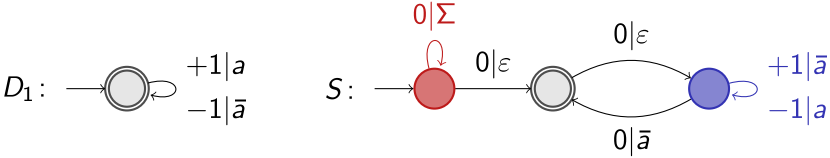

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If you really want to understand the relationships between language classes:

Study separability!

Appendix

Sources I

- Michael Blondin, Matthias Englert, Alain Finkel, Stefan Göller, Christoph Haase, Ranko Lazic, Pierre McKenzie, and Patrick Totzke.
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