α-β-Factorisation and the binary case of Simon's Congruence

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What is a Scattered Factor?

Example

palindrome

What is a Scattered Factor?

Example

palindrome

palm

What is a Scattered Factor?

Example

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and

What is a Scattered Factor?

Example

palindrome

palm

and

lime

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lime

dome

Scattered Factors

Definition.
A word $u = u[1] \cdots u[n] \in \Sigma^*$ scattered factor of $v \in \Sigma^*$ if

$$\exists x_1, \ldots, x_{n+1} \in \Sigma^* : \; v = x_1 u[1] x_2 u[2] \cdots x_n u[n] x_{n+1}$$

Simon's Congruence

Example

$\sim_k$

Simon's Congruence

Example

$\texttt{abaa} \sim_3 \texttt{abaaa}$?

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Simon's Congruence

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$\mathtt{abaa} \sim_3 \mathtt{abaaa}?$

$\mathrm{ScatFact}_3(\mathtt{abaa})$

$=$

$\{\mathtt{aaa}, \mathtt{aba}, \mathtt{baa}\}$

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$abaa \sim_3 abaaa?$

$$\mathrm{ScatFact}_3(abaa) = \{aaa, aba, baa\}$$

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Simon's Congruence

Definition.
The words u and v are Simon congruent modulo $k \in \mathbb{N}_0$ if

$$\mathrm{ScatFact}_\ell(u) = \mathrm{ScatFact}_\ell(v) \quad \text{for all } \ell \leq k$$

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- what is the index $|\Sigma^* / \sim_k|$ for a fixed $k \in \mathbb{N}$?

Scattered Factor Universality

Definition

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A word $w \in \Sigma^*$ is called k-universal if

$$\mathrm{ScatFact}_k(w) = \Sigma^k.$$

- $\iota(w)$ largest number such that $\mathrm{ScatFact}_k(w) = \Sigma^{\iota(w)}$

Arch Factorisation (Hébrard)

"Formalisation"

aabacbcbcabcabc

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# of Arches

$\iota(w) = 3$

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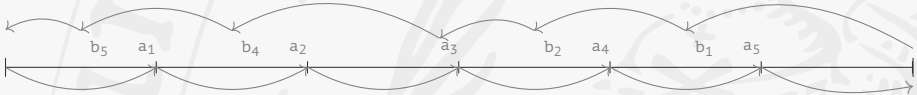
# of Arches	Rest	Modus
$\iota(w) = 3$	$\mathrm{r}(w) = \mathrm{bc}$	$\mathrm{m}(\mathrm{aabacbcbcabcabc}) = \mathrm{caa}$

α-β-Factorisation

both directions



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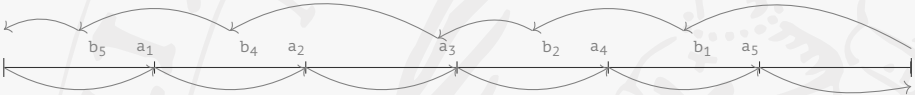
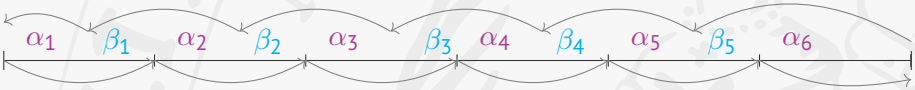


- arches and reverse arches always overlap

α-β-Factorisation

Refinement



α-β-Factorisation

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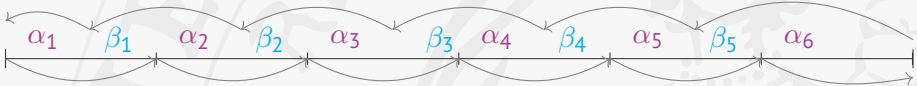

- first and last letter in β_i unique

- α_i to fill up arches with β_{i-1} and β_i
- letters in α_i arbitrarily often and permuted

α-β-Factorisation

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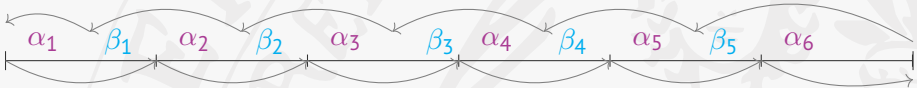

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Assume $\iota(w) < k$!

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key properties

Lemma.

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- up to $\sim_k$ factor of form $\alpha_i\beta_i \cdots \alpha_j$ can be replaced by a $(k - \iota(w) + (j - i))$-equivalent one

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freely exchange α bordered factors w.r.t. $\sim_{k-\ell}$ (ℓ β-factors *outside*)

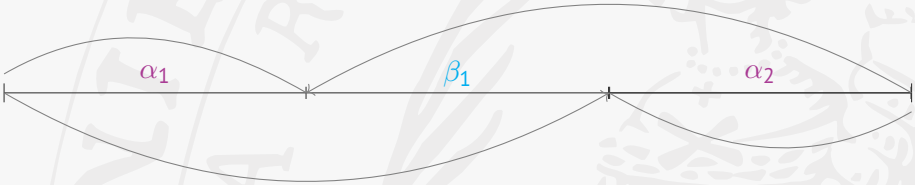
α-β-Decomposition

just triples

Theorem.
$w \sim_k \tilde{w} \Leftrightarrow \forall i \in [\iota(w)] : \alpha_i\beta_i\alpha_{i+1} \sim_{k-\iota(w)+1} \tilde{\alpha}_i\tilde{\beta}_i\tilde{\alpha}_{i+1}$

Binary Case

α-β-Factorisation



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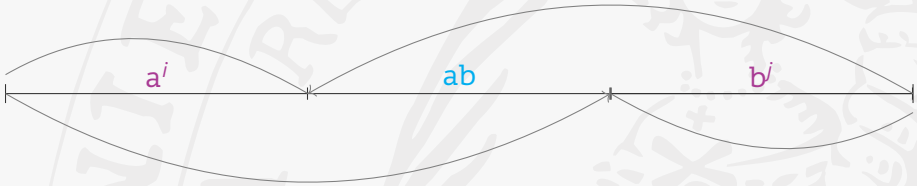

- $i, j \in \mathbb{N}_0$

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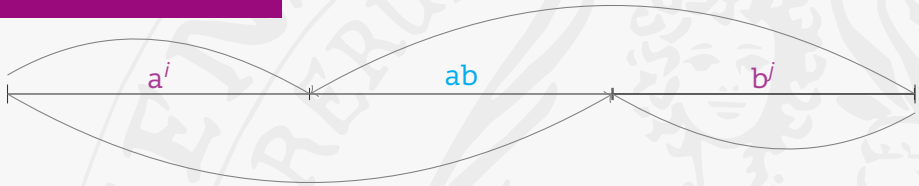

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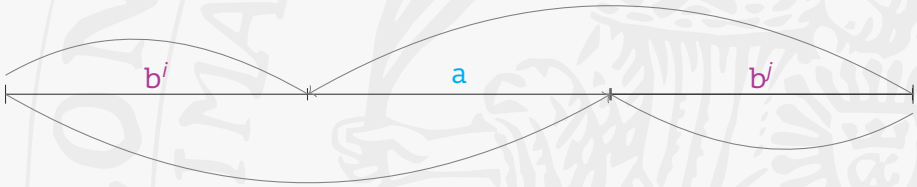

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- $\alpha_i \sim_{k-\iota(w)} \tilde{\alpha}_i$ for all $i \in [\iota(w) + 1]$
- $\beta_i = \tilde{\beta}_i$ for all $i \in [\iota(w)]$

Enumerate $|\Sigma^*/\sim_k|$

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- seperate by $\iota(w)$

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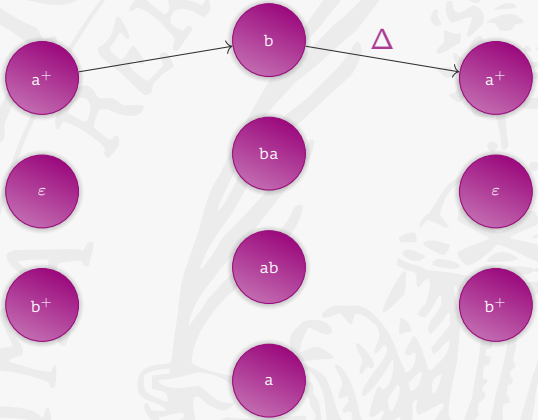
  - fix the difference $\Delta = k - \iota(w) \geq 1$

Shape of $\alpha\beta\alpha$ triples

Binary Case



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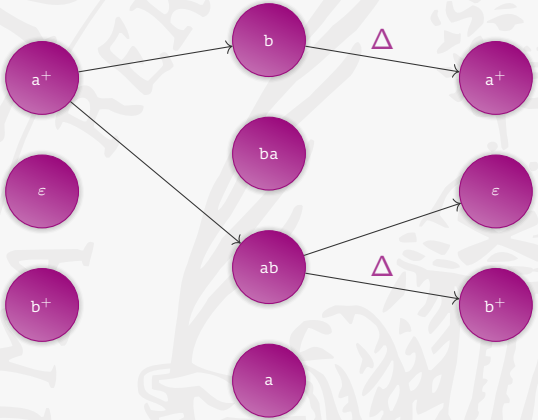

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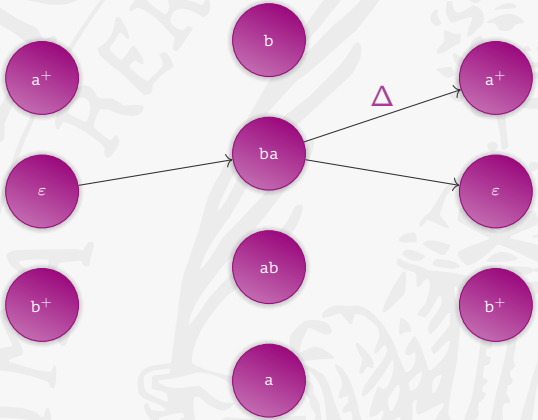

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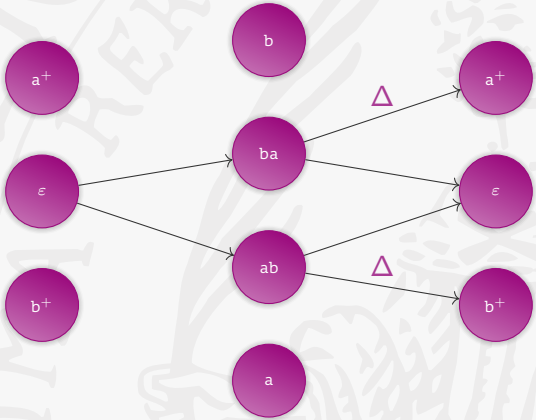

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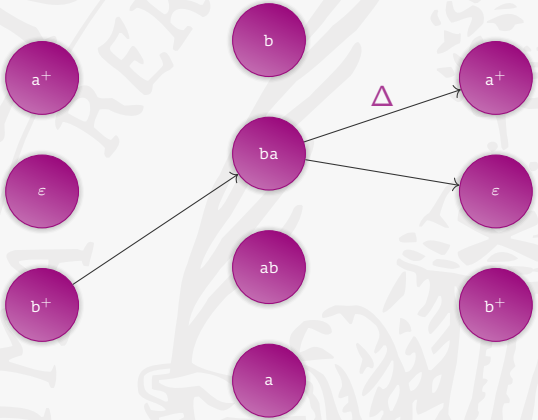

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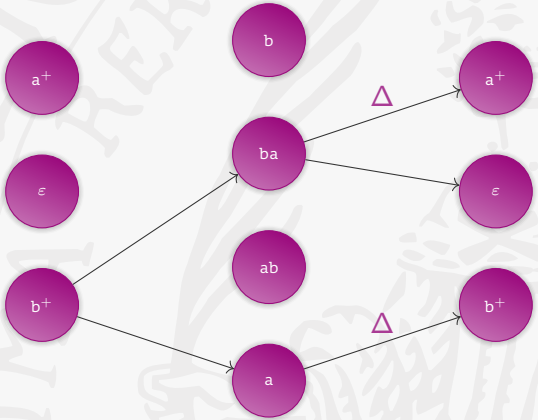

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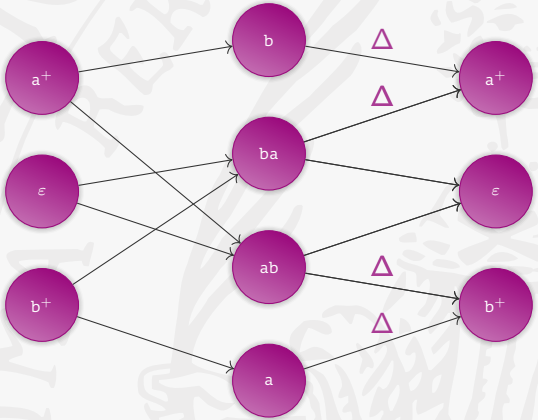

Enumeration of $|\Sigma^*/\sim_k$

Closed Formula

- number of classes of words with ℓ arches w.r.t. $\ell + \Delta$ ($\Delta = k - \iota(w)$)

$$c_\Delta(\ell) = |\{w \in \Sigma^* | \, \iota(w) = \ell\}/\sim_{\Delta+\ell}| = \left\| \begin{pmatrix} \Delta & \Delta & \Delta \\ 1 & 2 & 1 \\ \Delta & \Delta & \Delta \end{pmatrix}^{\ell} \cdot \begin{pmatrix} \Delta \\ 1 \\ \Delta \end{pmatrix} \right\|_1$$

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1, 4, 16, 68, 312, 1560, 8528, 50864, 329248, 2298592, 17203264, 137289920, 1162805376, 10409679744, 98146601216, 971532333824, 10068845515264, . . .

Ternary Alphabet

$\lvert alph(\alpha_0)\rvert, \lvert alph(\alpha_1)\rvert$	$alph(\alpha_0)$	$alph(\alpha_1)$	β RegExp
2,2	$\{a, b\}$ $\{a, b\}$	$\{a, c\}$ $\{a, b\}$	ba^*c c
2,1	$\{a, b\}$ $\{a, b\}$	$\{c\}$ $\{a\}$	$(ab^+ \mid ba^+)c$ ba^*c
2,0	$\{a, b\}$	$\emptyset$	$(ab^+ \mid b^+a)c$
1,1	$\{a\}$ $\{a\}$	$\{b\}$ $\{a\}$	$ab^+c \mid ac^+b \mid ca^+b$ ba^*c
1,0	$\{a\}$	$\emptyset$	$ba^*c \mid ab^+c$
0,0	$\emptyset$	$\emptyset$	ab^+c