Matching Patterns with Variables Under Simon's Congruence

Pamela Fleischmann, Sungmin Kim, Tore Koß, Florin Manea, Dirk Nowotka, Stefan Siemer, **Max Wiedenhöft**

What you will See

Introduction & Problems

Complexity Results

Conclusion

Words – Needed Terminology

Σ **Alphabet of constants** (of constant size σ)

$\Sigma := \{ \mathbf{a}, \mathbf{b}, \mathbf{n} \}$ $(\sigma = \mathbf{3})$

Σ^* **All words over alphabet**

$\mathbf{banana} \in \{ a, b, n \}^*$

$|w|$ **Length of a word**

$|banana| = \mathbf{6}$

ε **Empty word** $(|\varepsilon| = 0)$

$banana\boldsymbol{\varepsilon} = banana$

Subsequences

(Scattered Factors)

Subsequences

(Scattered Factors)

bananacakes

Subsequences

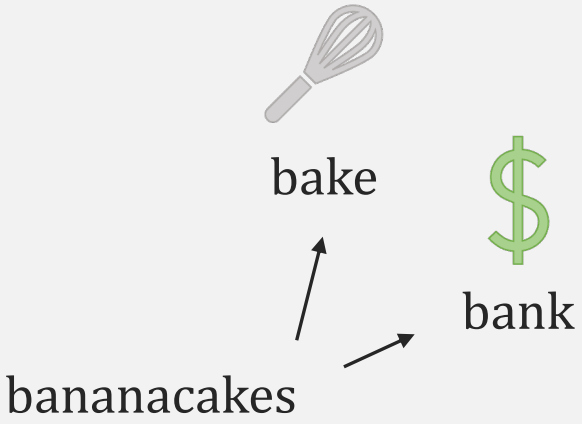
(Scattered Factors)

bake

bananacakes

Subsequences

(Scattered Factors)



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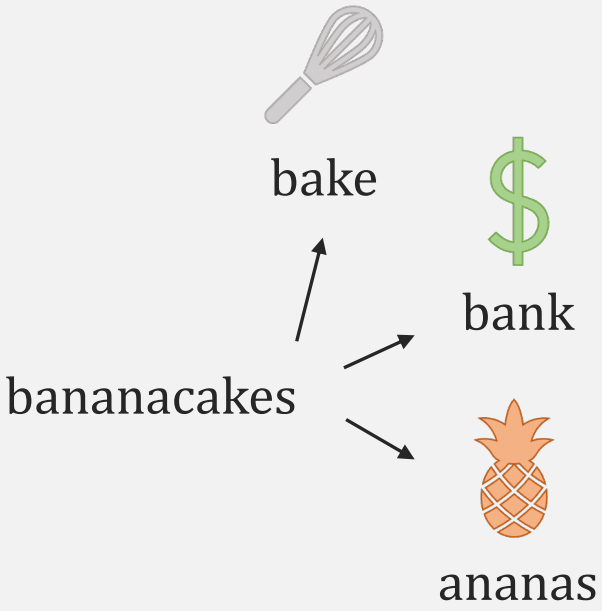

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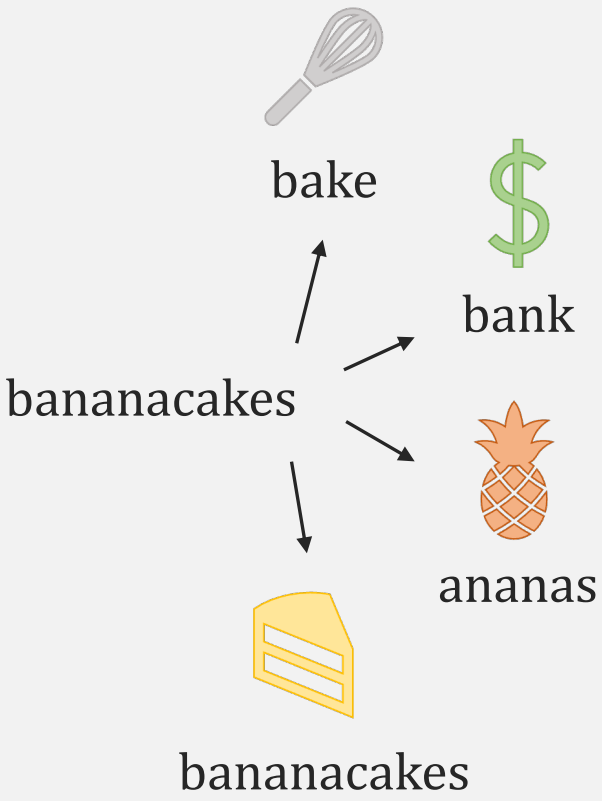

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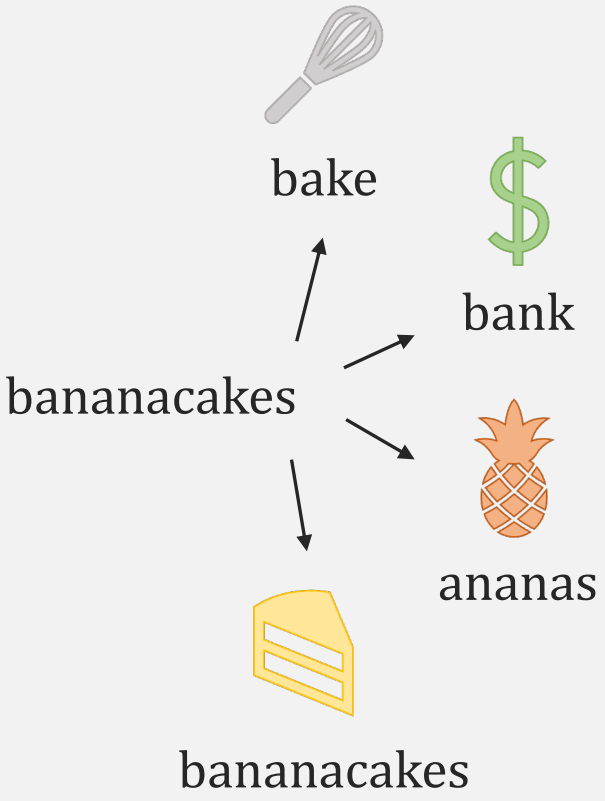

Notation

$\mathbb{S}(w)$ Set of all subsequences of w

$\mathbb{S}_k(w)$ Set of all subsequences of w up to length k

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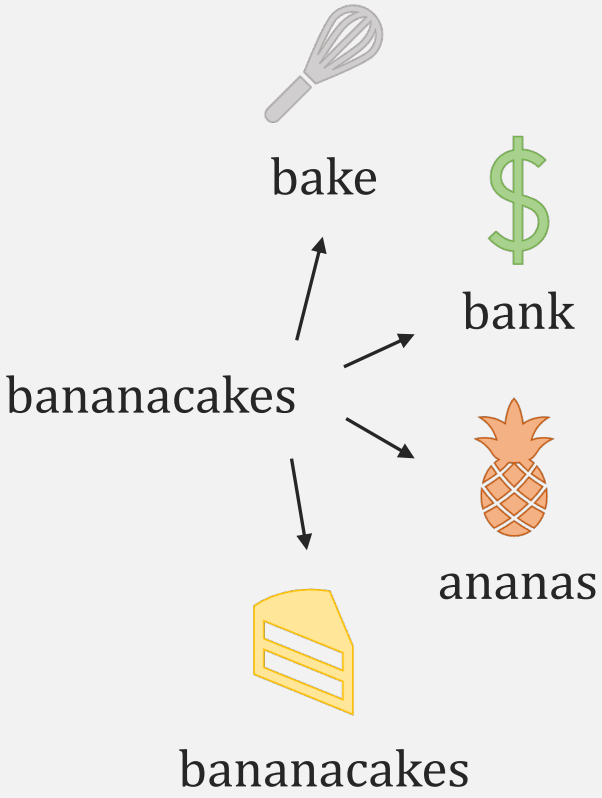

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k-Universality

w is k-universal ($\iota(w) = k$) iff $\mathbb{S}_k(w) = \Sigma^{\leq k}$

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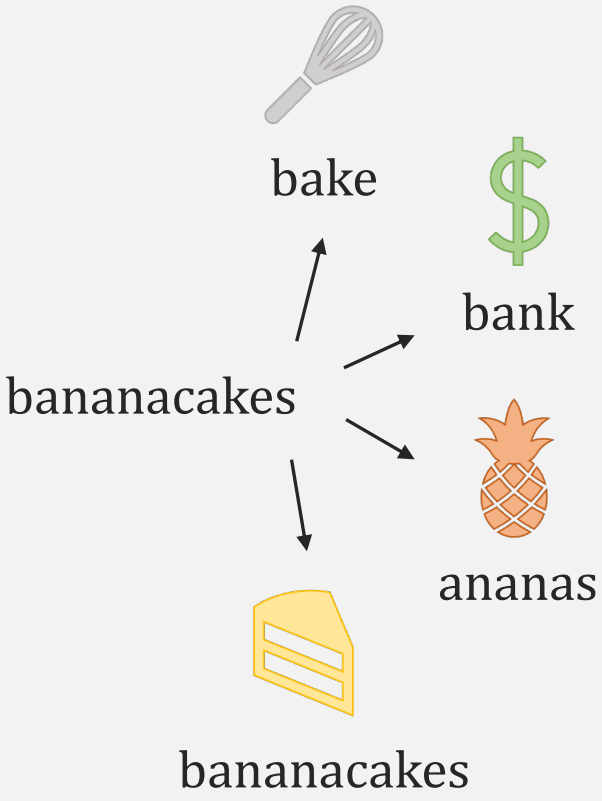

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Simon's Congruence

w_1 and w_2 are k-Simon congruent ($w_1 \sim_k w_2$) iff:

$$\mathbb{S}_k(w_1) = \mathbb{S}_k(w_2)$$

Patterns with Variables

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Infinite set of variables
(disjoint to Σ)

$$X = \{x_1, x_2, \ldots\}$$

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(over alphabet und variables)

$\mathrm{b}x_1x_1\mathrm{a}x_2 \in (\Sigma \cup X)^*$

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h

Substitutions
(map patterns to words)

$h : (\Sigma \cup X)^* \to \Sigma^*$

$\mathrm{b}\ x_1\ x_1\ \mathrm{a}\ x_2$ → $\mathrm{b\ an\ an\ a\ cake}$

$\mathrm{b\ an\ an\ a}\ \varepsilon$

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Fundamental Problem: Match

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Pattern α

Word w

Problem

Does there exist a substitution h with

$$h(\alpha) = w$$

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Match is NP-complete

Three Main Problems

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MatchUniv

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Can we find substitution h with

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WESimon

Given

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$\boldsymbol{h(\alpha) \sim_k h(\beta)}$

Let’s see how hard these Problems are!

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Example

$\Sigma = \{a, b, c\}$

$k = 3$

$\alpha = abc\ abc\ \boldsymbol{x_1}$

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Can we find substitution h with

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$\alpha = \text{abc abc } x_1 \longrightarrow \text{abc abc } \mathbf{abc}$

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$\Sigma = \{a, b, c, \mathbf{d}\}$?

$\alpha = \text{abc abc } x_1$

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MatchUniv is NP-Hard

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Reduce from 3CNFSAT!

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$(x_1 \vee \boldsymbol{x_2} \vee x_4) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \boldsymbol{\bar{x}_3})$

True with e.g. $\boldsymbol{x_2 = 1}$ and $\boldsymbol{x_3 = 0}$

1 MatchUniv is NP-Hard

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Reduction Layout

$\Rightarrow$

Map to MatchUniv instance
(with specific k)

$z_i\ u_i$

Two pattern variables z_i, u_i
(per clause variable $\mathrm{x_i}$)

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1010..

Binarisation Gadget

1 | 0

Boolean Gadget

$1 \leftrightarrow 0$

Complementation Gadget

$\wedge\vee$

Clause Gadget

MatchUniv is in NP

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Idea

Guess substitution h and check whether $\iota\big(h(\alpha)\big) = k$

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Find smaller representation for words with certain universality

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⇒ **Universality Signatures**

1 Subsequence Universality Signatures

(Schnoebelen & Veron 2023)

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<u>Compression of words using specific tuple</u>

preserves universality of words

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Concatenation Property (Schnoebelen & Veron 2023)

given universality signatures $s(w_1)$ and $s(w_2)$
$\Rightarrow s(w_1 w_2)$ computable in polynomial time

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Constant Signature Validity / Existence Check

given signature tuple
$\Rightarrow$ constant bound on word length to check if signature tuple exists
(if alphabet is of constant size)

NP-Containment by Verification

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Given

pattern α and number k

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Verification Approach

1: Guess signature tuple $\boldsymbol{s}(\boldsymbol{h}(\boldsymbol{x}))$ of potential substitution $\boldsymbol{h}$ for all $\boldsymbol{x} \in \mathbf{var}(\boldsymbol{\alpha})$ $O(|\mathrm{var}(\alpha)|)$

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MatchUniv is NP-complete!

Additional Results (under constant alphabet size)

- MatchUniv(α, k) in P if there exists unique variable in α
- MatchUniv(α, k) in P if the number of variables is bound by a constant

2

MatchSimon

2

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Can we find substitution h with

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Not strict! Solution trivial in many cases!

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Motivates to consider stricter version!

2

MatchStrictSimon

MatchStrictSimon

Given

Pattern α

Number k

Word w

Problem

Can we find substitution h with

$$h(\alpha) \sim_k w \;\&\; h(\alpha) \not\sim_{k+1} w$$

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Use same reduction approach used for MatchUniv

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- encode 3CNFSAT instance in pattern α similarly
- set k exactly the same way

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3CNFSAT instance solvable if and only if MatchStrictSimon(α, k, w) true

2

NP-Containment Sketch

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NP-Containment Sketch

MatchSimon

MatchStrictSimon

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NP-Containment Sketch

MatchSimon **MatchStrictSimon**

$$\mathbf{k} \geq |w|$$

NP-Containment Sketch

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Check if α matches w

NP-Containment Sketch

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$\mathbf{k} \geq \lvert w \rvert$	
Check if α matches w	Always answer no!

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Encode variable substitutions in words of maximum size	
$\boldsymbol{k}^{\boldsymbol{\sigma}}$	$(\boldsymbol{k}+\mathbf{1})^{\boldsymbol{\sigma}}$

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MatchSimon & MatchStrictSimon are NP-complete

For regular patterns MatchSimon and MatchStrictSimon are in P.

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WESimon

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Pattern β

Number k

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⇒ **WESimon is NP-hard**

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MatchSimon is particular case of **WESimon**

⇒ **WESimon is NP-hard**

In NP? Yes!

- If $k \leq |\alpha| + |\beta|$ then same as MatchSimon
- If $k > |\alpha| + |\beta|$ and $\beta \in \Sigma^*$ then instance of MatchSimon!
- If $k > |\alpha| + |\beta|$ and both contain variables then always yes!

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⇒ **WESimon is NP-complete!**

Again, not strict!

($h(\alpha) \sim_{k+1} h(\beta)$ allowed)

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WEStrictSimon

WEStrictSimon

Given

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Can we find substitution h with

$$\boldsymbol{h(\alpha) \sim_k h(\beta)} \;\&\; \boldsymbol{h(\alpha) \not\sim_{k+1} h(\beta)}$$

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Similar to MatchStrictSimon!

- NP-hardness: similar 3CNFSAT reduction
- NP-contaiment: as long as $k \leq |\alpha| + |\beta|$

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- NP-containment: as long as $k \leq |\alpha| + |\beta|$

NP-containment open for k without upper bound!

What have we seen?

Introduction & Problems

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2. **MatchSimon**
(strict and not strict variant) ⇒ both NP-complete

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2. **MatchSimon** (strict and not strict variant) ⇒ both NP-complete
3. **WESimon** (strict and not strict variant) ⇒ WESimon NP-complete
   ⇒ WEStrictSimon NP-complete for $k \leq |\alpha| + |\beta|$

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What are the parameterised complexities of these problems w.r.t.
→ the size of the alphabet?
→ the number of variables of the considered patterns?

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Thank You!

MatchUniv NP-Hardness

Final Assemblage of Gadgets

$$\alpha = \underbrace{\pi_{\#}\pi_{\$}}\ \underbrace{\pi_1^z\pi_1^u \dots \pi_n^z\pi_n^u}\ \underbrace{\xi_1 \dots \xi_n}\ \underbrace{\delta_1 \dots \delta_m}$$

Binarisation Gadgets

$h(z_i), h(u_i) \in \{1,0\}^*$

$\Rightarrow \boldsymbol{\iota(h(\pi_{\#}\pi_{\$}))} = \mathbf{2}$

Boolean Gadgets

$h(z_i), h(u_i) \in L(0^*|1^*)$

$\Rightarrow \boldsymbol{\iota(h(\pi_1^z\pi_1^u \dots \pi_n^z\pi_n^u))} = \mathbf{4n}$

Complementation Gadgets

$\boldsymbol{z_i} \in \{\mathbf{1}\}^+$ and $\boldsymbol{u_i} \in \{\mathbf{0}\}^+$

or

$\boldsymbol{z_i} \in \{\mathbf{0}\}^+$ and $\boldsymbol{u_i} \in \{\mathbf{1}\}^+$

$\Rightarrow \boldsymbol{\iota(h(\xi_1 \dots \xi_n))} = \mathbf{n}$

Clause Gadgets

$\forall j \in [m]: \iota\left(h\left(\delta_j\right)\right) = 1$

$\Rightarrow \boldsymbol{\iota(h(\delta_1 \dots \delta_m))} = \mathbf{m}$

3CNFSAT Instance solvable if and only if

$$\iota\big(h(\alpha)\big) = 5n + m + 2$$

(Otherwise, this universality can't be reached!)

MatchUniv is NP-Hard!

Universality Signatures

Subsequence Universality Signature $s(w)$

Consider word $w \in \Sigma^*$
Then $s(w) = (\gamma, \mathcal{K}, \mathcal{R})$ with

γ permutation of $\mathrm{alph}(w)$
(first occurrence of each letter in w)

$\mathcal{K}$ array of length $|\sigma|$
($\mathcal{K}[i]$ is equal to universality of suffix of w after first occurrence of $\gamma[i]$)

$\mathcal{R}$ array of length $|\sigma|$
($\mathcal{R}[i]$ contains alphabet of rest of suffix of w after first occurrence of $\gamma[i]$)

Example

$w = \mathbf{b}\ \mathbf{c}\ \mathrm{c}\ \mathbf{a}\ \mathrm{a}\ \mathrm{b}\ \mathrm{c}\ \mathrm{b}\ \mathrm{a}\ \mathrm{c}\ \mathrm{a}\ \mathrm{a}$

$\gamma = (\mathbf{b}, \mathbf{c}, \mathbf{a})$

$\mathcal{K} = (\mathbf{2}, \mathbf{2}, \mathbf{2})$

$\mathcal{R} = (\{\mathbf{a}, \mathbf{c}\}, \{\mathbf{a}, \mathbf{c}\}, \{\mathbf{a}\})$