Longest Common Subsequence with Gap Constraints

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Kaiserslautern

Introduction

Subsequence

w	b	a	c	b	a	a	b	a	d	a

Subsequence

ababa

d

bbb

ba

bc

w | b | a | c | b | a | a | b | a | d | a

baab

a

bb

c

aaaaa

b

Subsequence

ababa

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abc

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Subsequence

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Subsequences in TCS

Subsequences are a central concept in many different areas of TCS:

- Formal languages and logics (piecewise testable languages, subword order and downward closures).
- Combinatorics on words.
- Chemformatics.
- Modelling concurrency.
- Database theory (event stream processing).
- Algorithms (longest common subsequence, shortest common supersequence).

Subsequence embeddings

Σ is a finite alphabet, e. g., $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$.

	$w[1]$	$w[2]$	$w[3]$	$w[4]$	$w[5]$	$w[6]$	$w[7]$	$w[8]$
$w =$	a	b	b	a	c	b	a	b

$s =$	b	b	a
	$s[1]$	$s[2]$	$s[3]$

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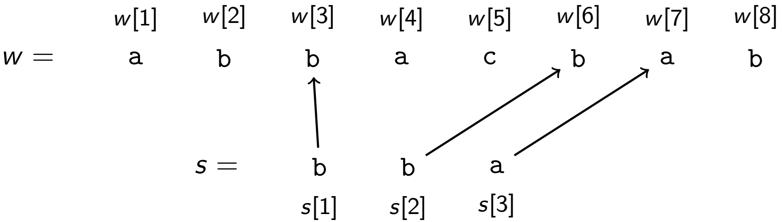

Notation

Embedding: $e : \{1, \ldots, |s|\} \to \{1, \ldots, |w|\}$ with $e(1) < \ldots < e(|s|)$.

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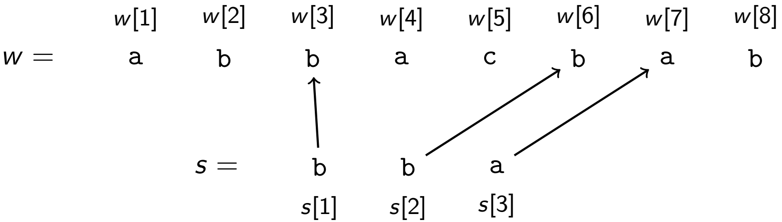

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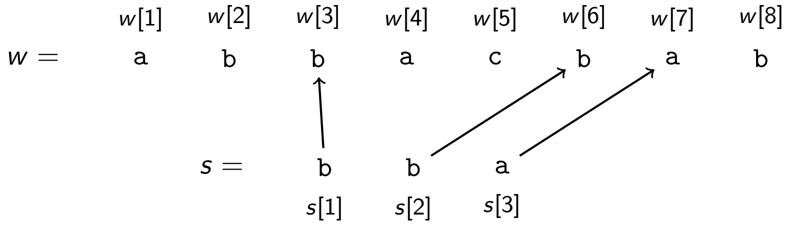

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s is subsequence of w ($s \preceq w$) if there is some embedding e with $s \preceq_e w$.

Subsequence Example

a b c b c a b c a b a c

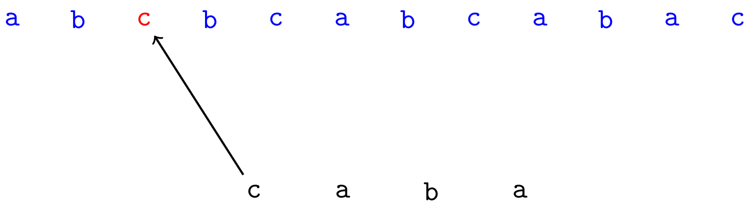
c a b a

Subsequence Example

a b c b c a b c a b a c

c a b a

Subsequence Example



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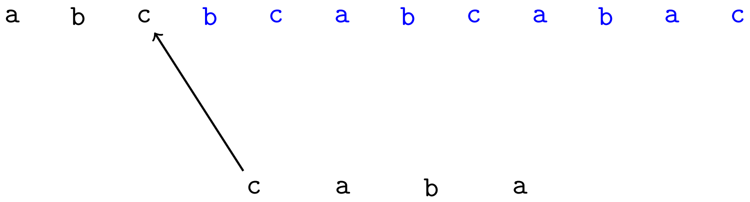

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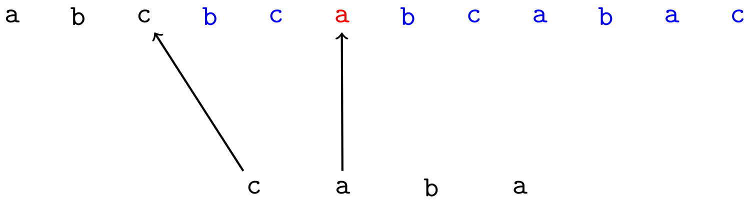

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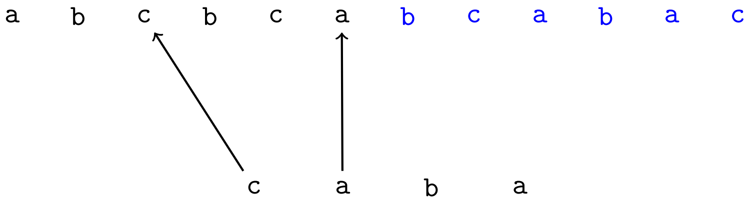

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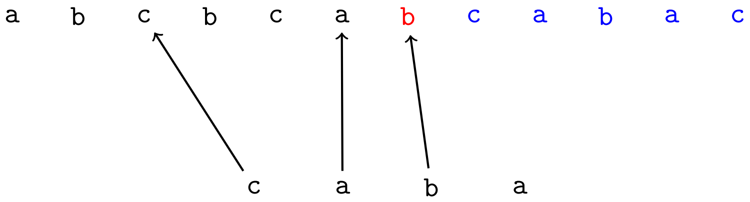

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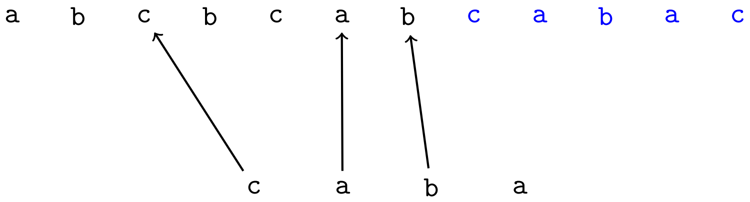

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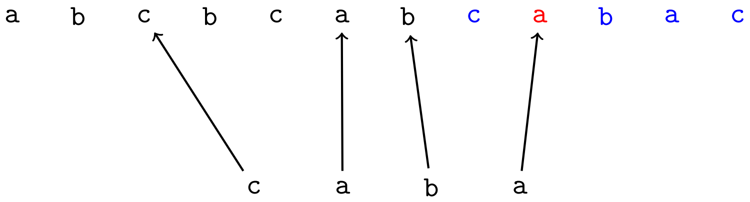

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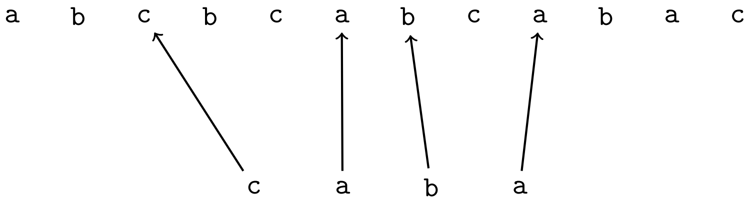

Adding Gap Constraints

Classic setting

Classical subsequences are usually considered with arbitrary embeddings.

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Constraint setting

Constraints on the embeddings of subsequences (Day, Kosche, Manea, Schmid ISAAC 22).

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For practical scenarios, it is reasonable to introduce *gap constraints*.
We restrict the length of the gaps by a lower and upper bound.

- Alignments of bio-sequences.
- Modelling single processor scheduling with fairness properties.

One could also restrict the letters/languages of the gaps.

- Complex event processing $\rightarrow$ forbidding events in specific positions of a subsequence (not in this paper).

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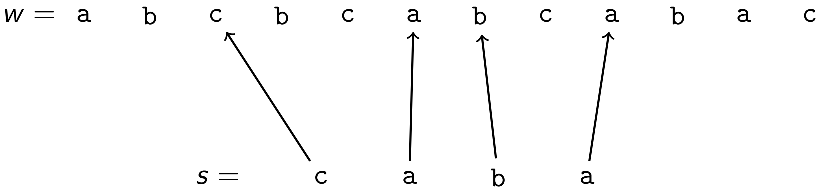
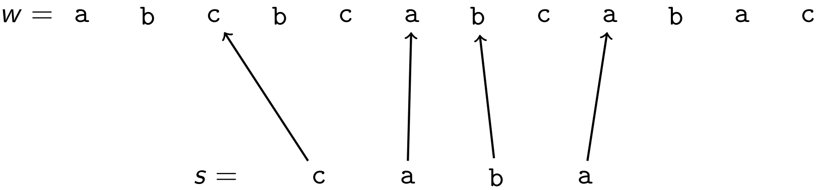
$s =$ c a b a



$$\mathrm{gap}_e(w,i) = w[e(i)+1..e(i+1)-1] \qquad i \in \{1,\ldots,|w|-1\}$$



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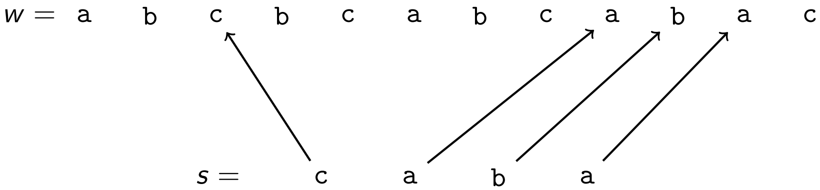

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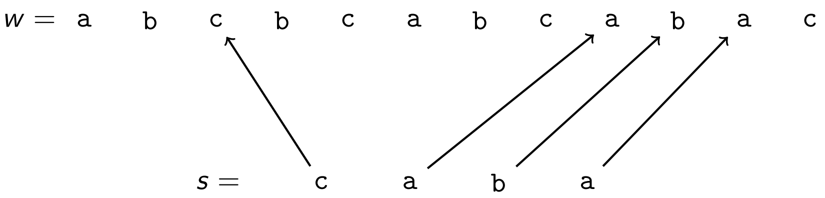

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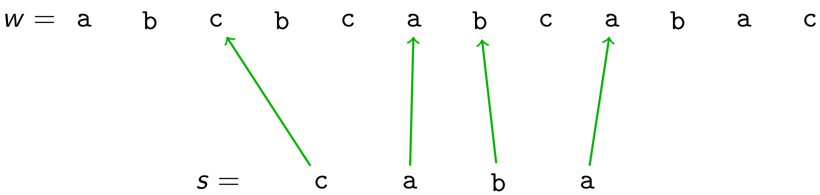
Gap constraints

$gc = (C_1, \ldots, C_{|s|-1})$, where $C_i = (\ell_i, u_i)$ for every $i \in \{1, \ldots, |s| - 1\}$.
(tuple of gap constraints)
The embedding e *satisfies* gc *w.r.t.* s, if, for every $i \in [|s| - 1]$, $\ell_i \leq |\mathrm{gap}_e(w, i)| \leq u_i$.



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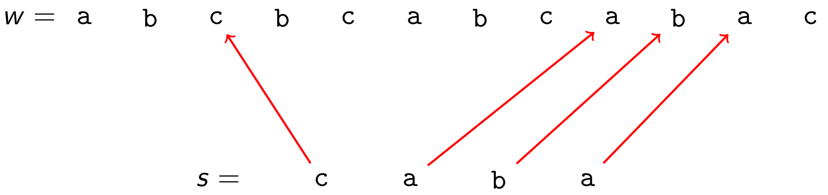

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LCS Problem

Problem

Given v, w of size m and n, compute the largest $k \in [m]$ such that there exists a common subsequence s of both v and w with $|s| = k$.

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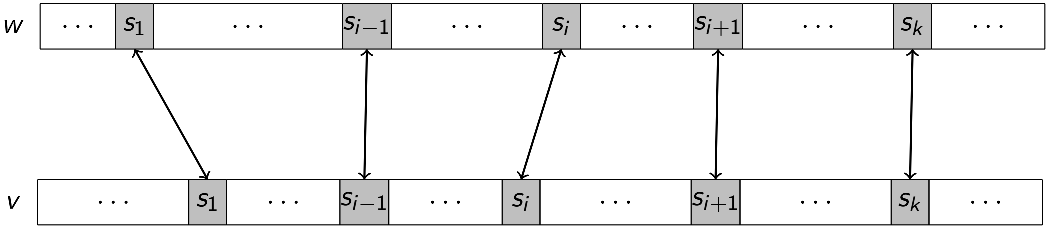

LCS Result

Idea

Classical dynamic programming approach.

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Algorithm and lower bound (Abboud, Backurs, Williams)

Folklore algorithm solving LCS in $O(N)$ time ($N = mn$). Conditional lower bound as a $O(N^{1-\epsilon})$ would refute SETH.

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Problem: We want to add gap constraints, but we dont know how long the LCS is.

Consider various scenarios:

- Given a large enough tuple of constraints in the input.
- All gaps have the same constraint.
- Draw from a constant sized pool of constraints.
- Gap constraints are determined by surrounding letters.

Problems

LCS-MC

Problem

Given $v, w \in \Sigma^*$ and an $(m-1)$-tuple of gap-length constraints gc, compute the largest $k \in \mathbb{N}$ such that there exists a common $gc[1:k-1]$-subsequence s of v and w, with $|s| = k$.

LCS is a particular case of $\mathrm{LCS\text{-}MC}$, where $gc = ((0, n), \ldots, (0, n))$.

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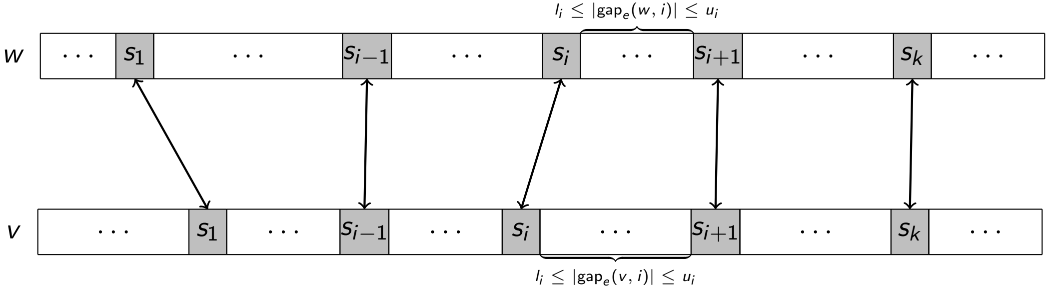

LCS-MC Results

Idea

- for each $p \in [m]$, a matrix $M_p \in \{0,1\}^{m \times n}$
- $M_p[i,j] = 1$ if and only if there exists a string s with $|s| = p$ ending in $v[i]$ and $w[j]$ satisfying $gc[1 : p-1]$.
- compute M_1 by setting $M_1[i,j] = 1$ if and only if $v[i] = w[j]$
- $M_p[i][j] = 1$, iff $M_{p-1}[I][J]$ contains a 1 and $v[i] = w[j]$.
- $M_{k+1}[\cdot][\cdot] = 0$, while $M_k[i][j] = 1$ for some i, j.

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Result

$\mathrm{LCS\text{-}MC}$ can be solved in $O(Nk)$ time, where k is the largest number for which there exists a common $gc[1 : k - 1]$-subsequence s of v and w.

LCS-MC-INC Problem

Problem

Consider the variation of $\mathrm{LCS\text{-}MC}$ for increasing tuples of gap-length constraints gc; this variant is called $\mathrm{LCS\text{-}MC\text{-}INC}$.

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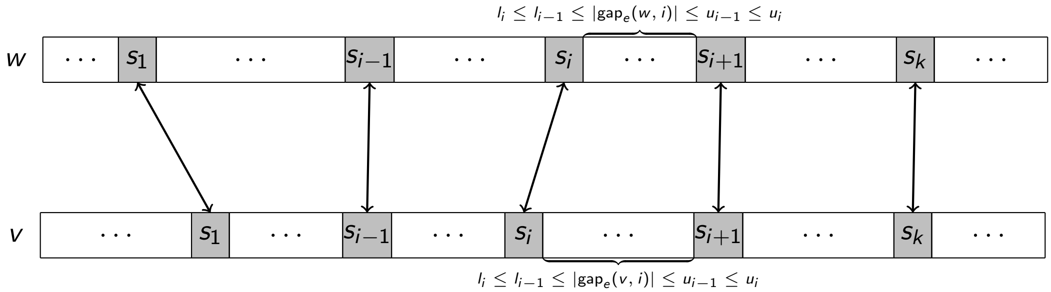

LCS-MC-INC Results

Idea

- Matrix $M[i][j] = p$ if there is subsequence s with $|s| = p$ ending in $v[i]$, $w[j]$ satisfying $gc[1 : p-1]$.
- Using a 2D-Segment tree data structure for efficient maximum queries..

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Result

LCS-MC-INC can be solved in $O(N \log^2 N)$.

LCS-1C Problem

Problem

Given $v, w \in \Sigma^*$ and an $(m-1)$-tuple of identical gap-length constraints $gc = ((\ell, u), \ldots, (\ell, u))$, compute the largest $k \in \mathbb{N}$ such that there exists a common $gc[1 : k-1]$-subsequence s of v and w, with $|s| = k$.

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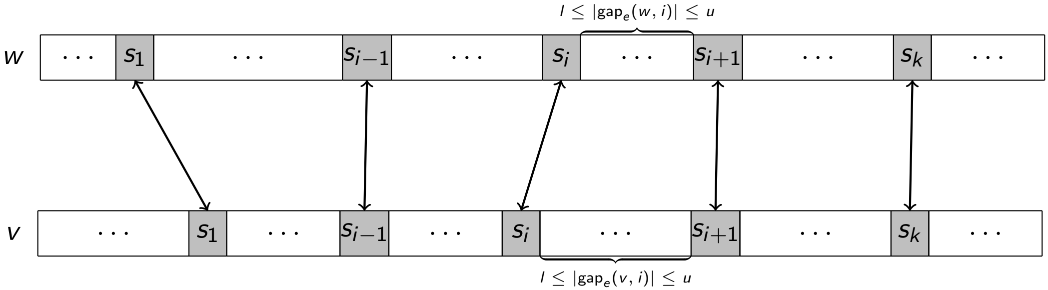

LCS-1C Results

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- Matrix $M[i][j] = p$ if there is subsequence s with $|s| = p$ ending in $v[i]$, $w[j]$ satisfying $gc[1 : p-1]$ where $gc[i] = gc[i+1]$.
- Use maximum queues on columns to report maximum of relevant part within each column.
- Use maximum queue on each row that gets maxima of each column as input.

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Result

LCS-1C can be solved in $O(N)$ time.

LCS-Σ Problem

Problem

Given two words $v, w \in \Sigma^*$ and two functions $left : \Sigma \to [n] \times [n]$ and $right : \Sigma \to [n] \times [n]$, compute the largest number $k \in \mathbb{N}$ such that there exists a common $(left, right)$-subsequence s of v and w, with $|s| = k$.

When $left(a) = (0, n)$ for all $a \in \Sigma$ (respectively, $right(a) = (0, n)$ for all $a \in \Sigma$), the gap constraints are defined only by the function *right* (respectively, *left*), and the problem $\mathrm{LCS} - \Sigma$ is denoted $\mathrm{LCS} - \Sigma\mathrm{R}$ (respectively, $\mathrm{LCS} - \Sigma\mathrm{L}$).

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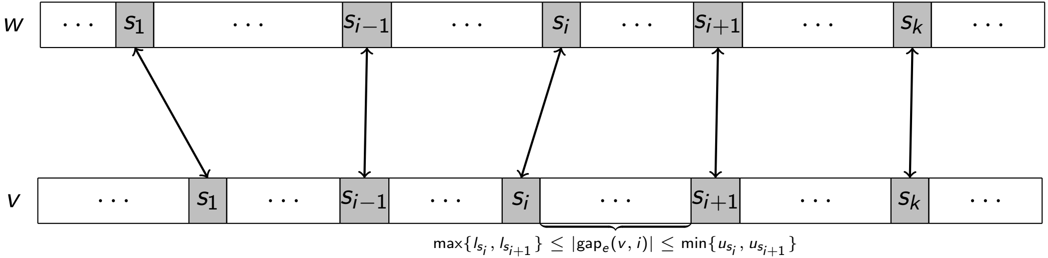

LCS-Σ Results

$|\Sigma| = \sigma$

Idea LCS-ΣR

- maintain σ many submatrices.
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LCS-Σ

- maintain σ^2 many submatrices.
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LCS-Σ Results

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- maintain σ many submatrices.
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LCS-Σ

- maintain σ^2 many submatrices.
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Result

$\mathrm{LCS}\text{-}\Sigma\mathrm{R}$, $\mathrm{LCS}\text{-}\Sigma\mathrm{L}$ can be solved in $O(\min\{N\sigma, N \log m\})$.
$\mathrm{LCS}\text{-}\Sigma$ can be solved in $O(\min\{N\sigma^2, N\sigma \log m\})$ time.

Conclusion

Summary

- LCS-MC can be solved in $O(Nk)$ time.
- LCS-MC-INC can be solved in $O(N \log^2 N)$.
- LCS-1C can be solved in $\mathrm{O}(N)$ time.
- LCS-O(1)C-SYNC can be solved in $\mathrm{O}(N)$ time.
- LCS-Σ can be solved in $O(\min\{N\sigma^2, N\sigma \log m\})$ time.
- LCS-ΣR, LCS-ΣL can be solved in $O(\min\{N\sigma, N \log m\})$.
- LCS-BR can be solved in $O(NB^{o(1)})$time.

Outlook

- Can the results of $\mathrm{LCS\text{-}MC}$ be improved? (or lower bounds)
- Improving Σ dependancy in $\mathrm{LCS} - \Sigma$?
- Other constraints? (e.g. Regular language constraints).
- Efficiently computing the actual longest common constrained subsequence.
- Different constraints on embeddings in v and w.
- Consider any subsequence problem in the context of gap constraints.

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Thank you!